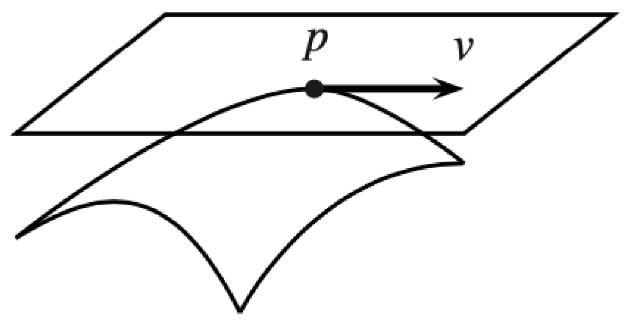
Tangent Vectors as Derivations

Christian Reynaldo Mentor: Daniel Halmrast

University of California, Santa Barbara

Introduction

In elementary calculus, the tangent space is typically introduced as the vector space orthogonal to the gradient of a function at a point. Intuitively, one can visualize any tangent vector as an arrow emanating from the given point,



typically satisfying an equation of tangency. This approach works well when considering surfaces given by a single smooth function because we can imagine how a surface might sit in $\mathbb{R}^3$. However, on the more abstract subject of manifolds we aim for a more intrinsic definition of the tangent space.

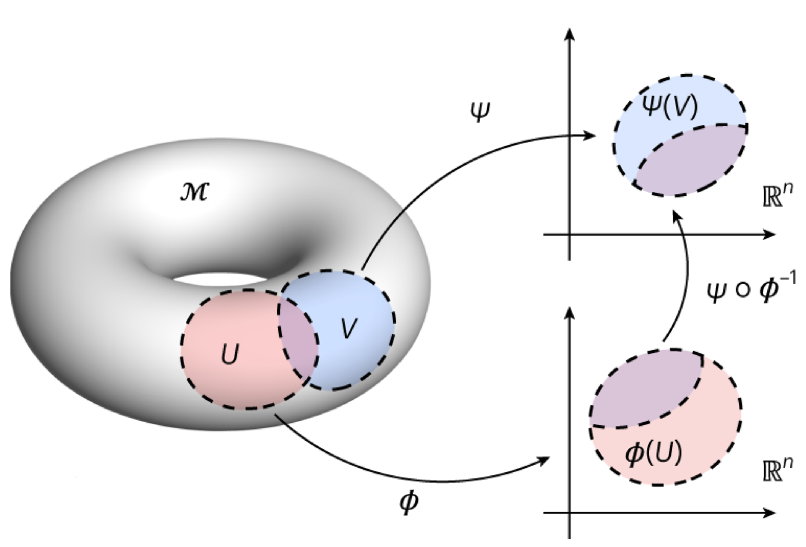
Definitions

Germ: A germ at $p \in \mathbb{R}^n$ is an equivalence class of C^∞ real-valued functions wherein two functions are equivalent if they agree on some neighborhood of p. In this way any directional derivative can be thought to operate on the set of germs at p. This set is an algebra over $\mathbb{R}$ denoted by C_p^∞.

Derivation: A derivation is a linear map $D : C_p^\infty \to \mathbb{R}$ satisfying the Leibniz rule $D(fg) = (Df)g(p) + f(p)Dg$. The set of all derivations of this kind is the real vector space $\mathcal{D}_p\mathbb{R}^n$.

Coordinate Chart: A topological space M is locally Euclidean of dimension n if every point $p \in M$ has a neighborhood U such that there is a homeomorphism ϕ from U into an open subset of $\mathbb{R}^n$. The pair (U, ϕ) is called a coordinate chart.

Smooth Manifold: A topological space M is said to be a smooth manifold if it is Hausdorff, second countable, and has a C^∞ atlas. An atlas is a collection of coordinate charts that cover M, and we call it



C^∞ if the transition functions are smooth. Some classical examples include the n-sphere, the torus, and perhaps the most elementary is $\mathbb{R}^n$ itself.

Smooth Map: A map of manifolds $F : M \to N$ is said to be smooth at $p \in M$ if, for coordinate charts (U, ϕ) containing p and (V, ψ) containing $F(p)$, we have $\psi \circ F \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ being smooth at $\phi(p) \in \mathbb{R}^m$.

Tangent Vectors as Derivations in $\mathbb{R}^n$

In an intuitive sense tangent vectors might best be thought of as directions of travel. For this reason a tangent vector to $p \in \mathbb{R}^n$ is any n-dimensional vector $v = \langle v_1, ..., v_n \rangle$. The set of tangent vectors forms a vector space $T_p\mathbb{R}^n$. For any tangent vector v at p, the directional derivative $D_v : C_p^\infty \to \mathbb{R}$ is linear and satisfies the Leibniz rule. Hence, it is in fact a derivation.

Theorem

The map $\varphi : T_p\mathbb{R}^n \to \mathcal{D}_p\mathbb{R}^n$ given by $v \mapsto D_v$ is an isomorphism of vector spaces.

Proof

Injectivity: Suppose $D_v = 0$ for some $v \in T_p\mathbb{R}^n$. If we apply D_v to the coordinate function r^j then we have

$$0 = D_v(r^j) = \sum_i v^i \frac{\partial}{\partial r^i}\bigg|_p r^j = v^j$$

Since this is true for $1 \le j \le n$ we have $v = 0$, and so φ is injective.

Surjectivity: Let D be an arbitrary derivation at p, and let $f : \mathbb{R}^n \to \mathbb{R}$ be the representation of some germ in C_p^∞. By Taylor's theorem with remainder there exists smooth functions $g_i(x)$ in a neighborhood of p such that

$$f(x) = f(p) + \sum (r^i(x) - p^i)g_i(x), \quad g_i(p) = \frac{\partial f}{\partial r^i}(p)$$

Now, applying D to both sides we get by the Leibniz rule,

$$Df = \sum \left(Dr^i g_i(p) + (p^i - p^i)Dg_i\right) = \sum (Dr^i)\frac{\partial f}{\partial r^i}(p)$$

$$D = \sum (Dr^i)\frac{\partial}{\partial r^i}\bigg|_p$$

Thus $D = D_v$ where $v = \langle Dr^1, ..., Dr^n \rangle$. This shows that every derivation is the directional derivative with respect to some vector, and so φ is a bijection.

With this in mind, we will redefine a tangent vector at p in $\mathbb{R}^n$ to be a derivation at p, and the tangent space $T_p\mathbb{R}^n$ is the vector space of derivations with basis $\{\partial/\partial r^i|_p\}_{i=1}^n$.

Generalizing to Manifolds

It is rather straightforward now to extend our idea of a tangent space to manifolds. We simply tweak our derivation definition to be a map $D : C_p^\infty(M) \to \mathbb{R}$, where $C_p^\infty(M)$ denotes the set of germs at any $p \in M$. These derivations form the tangent space T_pM, and the above becomes the particular case $M = \mathbb{R}^n$.

Our goal has thus been reached, as the tangent space has been defined in a way that does not depend on any coordinate chart. However, each coordinate chart (U, ϕ) containing p can yield a basis for T_pM as follows. We define the derivation $\partial/\partial x^i|_p$ such that for any $f \in C_p^\infty(M)$ we have

$$\frac{\partial}{\partial x^i}\bigg|_p f = \frac{\partial}{\partial r^i}\bigg|_{\phi(p)} (f \circ \phi^{-1})$$

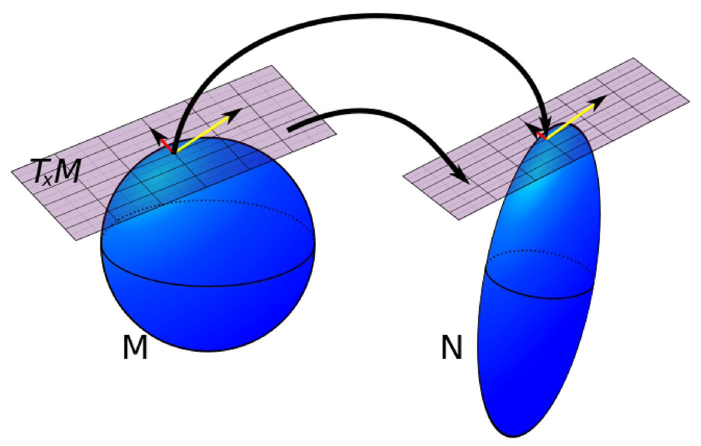
The collection of these derivations are linearly independent, and hence form a basis.

References

[1] Loring W. Tu. Manifolds. *An Introduction to Manifolds.* Springer, 2nd edition, 2011.

The Pushforward

Given a smooth map $F : M \to N$, the pushforward of F at $p \in M$ is a linear map $F_* : T_pM \to T_{F(p)}N$ such that for any $v \in T_pM$ and $f \in C_{F(p)}^\infty(N)$ we have $F_*(v)f = v(f \circ F)$.



Any coordinate chart inverse ϕ^{-1} provides a smooth map of manifolds between an open subset of $\mathbb{R}^n$ and a manifold M. For a point $p \in M$ we can then consider the pushforward $(\phi^{-1})_* : T_{\phi(p)}\mathbb{R}^n \to T_pM$. If we attempt to apply this map to our basis vectors $\partial/\partial r^i|_p$ we get the following result.

$$(\phi^{-1})_*\left(\frac{\partial}{\partial r^i}\bigg|_{\phi(p)}\right) f = \frac{\partial}{\partial r^i}\bigg|_{\phi(p)} (f \circ \phi^{-1}) = \frac{\partial}{\partial x^i}\bigg|_p f$$

Properties

Matrix Representation: Being linear, the pushforward can be represented by a matrix. This matrix is the Jacobian $[\partial F^i/\partial x^j(p)]$.

The Chain Rule: One final property of the pushforward that will be used in the next section is its chain rule. Some elementary linear algebra gives us the following powerful result: If $F : M \to N$ and $G : N \to P$ are both smooth maps of manifolds, then we have $(G \circ F)_* = G_* \circ F_*$.

Applications to Calculus

The usual chain rule taught in calculus can be proven as a particular case for when we consider smooth maps from $\mathbb{R}^m$ to $\mathbb{R}^n$.

As an example, let $F : \mathbb{R} \to \mathbb{R}^3$ and $G : \mathbb{R}^3 \to \mathbb{R}$ be smooth functions and let w be such that

$$w = (G \circ F)(t) = G\big(F^1(t), F^2(t), F^3(t)\big)$$

The pushforwards F_*, G_*, and $(G \circ F)_*$ are given by the following matrices.

$$F_* = \begin{bmatrix} dF^1/dt \\ dF^2/dt \\ dF^3/dt \end{bmatrix} \quad G_* = \begin{bmatrix} \frac{\partial w}{\partial F^1} & \frac{\partial w}{\partial F^2} & \frac{\partial w}{\partial F^3} \end{bmatrix} \quad (G \circ F)_* = \frac{dw}{dt}$$

The chain rule for the pushforward gives us $(G \circ F)_* = G_* \circ F_*$, or equivalently through multiplication of the matrices above,

$$\frac{dw}{dt} = \frac{\partial w}{\partial F^1}\frac{dF^1}{dt} + \frac{\partial w}{\partial F^2}\frac{dF^2}{dt} + \frac{\partial w}{\partial F^3}\frac{dF^3}{dt}$$