PARABOLIC SMOOTHING FOR LINEAR HEAT EQUATION

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1. Introduction

Solutions to the inital value problem (IVP) for the linear heat equation

$$\begin{cases} \partial_t u = \partial_x^2 u, \quad x, t \in \mathbb{R}, \\ u(x,0) = u_0(x) \end{cases} \tag{1.1}$$

heat

exhibit a very strong smoothing effect. Using harmonic analysis or the energy method it will be shown that when the initial data lies in $L^2(\mathbb{R})$, the solution possesses (for positive times) classical derivatives of all orders which also lie in $L^2(\mathbb{R})$. This is the so-called *parabolic smoothing effect*.

Following [1], the IVP (1.1) is said to be *locally well-posed* in the Banach space X if to every $u_0 \in X$, there exists $T > 0$ and a unique solution $u = u(x,t)$ satisfying

$$u \in C([0,T] : X). \tag{1.2}$$

persist

Furthermore, the solution map $u_0 \mapsto u$ from X into $C([0,T] : X)$ is continuous. If T can be taken arbitrarily large the IVP is said to be *globally well-posed*. The condition (1.2) is called the persistence property and when it holds the solution curve $t \mapsto u(\cdot, t)$ is said to describe a dynamical system on X.

We have the following local well-posedness theorem for the IVP (1.1).

Theorem A. *Suppose $u_0 \in L^2(\mathbb{R})$. Then for every $T > 0$ there exists a unique solution $u = u(x,t)$ to the IVP (1.1) satisfying*

$$u \in C([0,T] : L^2(\mathbb{R})) \tag{1.3}$$

with the function $t \mapsto \|u(\cdot,t)\|_2$ nonincreasing. Moreover, the solution depends continuously on the initial data.

The space $L^2(\mathbb{R})$ can be replaced with $H^s(\mathbb{R})$ for any $s \geq 0$.

Recall that Fourier analysis provides an explicit representation for the solution to (1.1) via the formula

$$u(x,t) = [\exp(-4\pi^2 t|\xi|^2)\hat{u}_0]^\vee \tag{1.4}$$

solution

or, alternatively $u = K_t * u_0$ where

$$K_t(x) = \frac{e^{-|x|^2/4t}}{\sqrt{4\pi t}} \tag{1.5}$$

is called the heat kernel.

The statement of the parabolic smoothing effect described here requires the definition of the L^2-based Sobolev spaces. Define the homogeneous derivative D and its inhomogeneous counterpart J by the Fourier multipliers

$$\widehat{D^s f}(\xi) = |\xi|^s \hat{f}(\xi) \qquad \text{and} \qquad \widehat{J^s f}(\xi) = \langle \xi \rangle^s \hat{f}(\xi), \quad s \in \mathbb{R}, \tag{1.6}$$

where $\langle x\rangle = (1+x^2)^{1/2}$. Then for $s \geq 0$ we have

$$H^s(\mathbb{R}) := \left\{ f \in L^2(\mathbb{R}) : \int_{-\infty}^{\infty} (1+\xi^2)^s |\hat{f}(\xi)|^2 \, d\xi < \infty \right\} \tag{1.7}$$

with norm

$$\|f\|_{H^s} = \|J^s f\|_2 \sim \|f\|_2 + \|D^s f\|_2. \tag{1.8}$$

Theorem 1. *(Parabolic Smoothing) Suppose $u_0 \in L^2(\mathbb{R})$ and let u be the solution to* (1.1) *provided by Theorem A. Then for any $t > 0$ and $k \in \mathbb{Z}^+$*

$$u(\cdot, t) \in H^k(\mathbb{R}). \tag{1.9}$$

Since local well-posedness ensures the L^2-norm of the solution is bounded for positive times, it suffices to control $\|\partial_x^k u(\cdot,t)\|_2$ for $t > 0$ and k a positive integer.

The Sobolev embedding states that if $f \in H^{k+1}(\mathbb{R})$, then $\partial_x^k f$ is continuous and bounded. Therefore $u(\cdot,t) \in C_b^\infty(\mathbb{R})$ for positive times. To see that this result is not time reversible, consider initial data $u_0 \in L^2(\mathbb{R})$ which is not continuous (thus is not $H^1(\mathbb{R})$). At $t = 1$, the solution is smooth and so the solution can lose regularity in finite time when time runs in reverse.

Intuition for the parabolic smoothing effect can be gained by considering the special solution

$$u(x,t) = e^{-\lambda^2 t} \cos(\lambda x) \tag{1.10}$$

which demonstrates that energy present in the intial data with a particular frequency λ is damped in time by a factor exponential in λ^2; higher frequencies decay more quickly. As the Sobolev spaces capture differentiability in terms of the decay of a function in frequency space, one expects a gain in regularity.

2. Smoothing via Fourier Analysis

We compute this norm explicitly using the representation (1.4). Recall that when differentiating the solution, the derivative falls on the exponential

$$\begin{aligned}
\partial_x u(x,t) &= \partial_x \int_{-\infty}^{\infty} e^{-4\pi^2 t|\xi|^2} e^{-2\pi i x\xi} \hat{u}_0(\xi) \, d\xi \\
&= \int_{-\infty}^{\infty} (2\pi i\xi) e^{-4\pi^2 t|\xi|^2} e^{-2\pi i x\xi} \hat{u}_0(\xi) \, d\xi,
\end{aligned}$$

and so

$$\partial_x^k u(x,t) = \int_{-\infty}^{\infty} (2\pi i\xi)^k e^{-4\pi^2 t|\xi|^2} e^{-2\pi i x\xi} \hat{u}_0(\xi) \, d\xi.$$

Using Parseval's identity

$$\begin{aligned}
\|\partial_x^k u(\cdot,t)\|_2 &= \|(2\pi i\xi)^k \exp(-4\pi|\xi|^2 t)\hat{u}_0\|_2 \\
&\leq (2\pi)^k \||\xi|^k \exp(-4\pi|\xi|^2 t)\hat{u}_0\|_2 \\
&\leq c_k \|u_0\|_2
\end{aligned}$$

since for any $k \in \mathbb{Z}^+$, $c > 0$,

$$\sup_{x\in\mathbb{R}} |x^k e^{-cx^2}| < \infty.$$

The result follows from the exponential decay of the heat kernel.

3. Smoothing via Convolution

A second proof of the parabolic smoothing effect follows from the convolution representation of the solution, properties of convolution and the following theorem.

Theorem B. *(Young's Inequality) Let $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ with $1 \le p, q, r \le \infty$. Then*

$$\|f * g\|_r \le \|f\|_p \|g\|_q.$$

Recall that $u = K_t * u_0$ so that

$$\partial_x^k u(x,t) = (\partial_x^k K_t * u_0)(x).$$

Applying Young's inequality with $p = 1$ and $r = q = 2$ produces

$$\|\partial_x^k u(\cdot,t)\|_2 \le \|\partial_x^k K_t\|_1 \|u_0\|_2$$

for $k \in \mathbb{Z}^+$ and $t > 0$. Note that $\partial_x^k K_t$ has the form $p(x)K_t(x)$ for p a polynomial in x. Thus $\|\partial_x^k K_t\|_1 < \infty$ due to the exponential decay of the heat kernel and the result follows. The choices of p, q, r are not essential to the arugment; Young's inequality generates a wide range of estimates for solutions to the heat equation.

4. Smoothing via the Energy Method

This technique is, perhaps, the most involved of the three detailed here, but it can be modified to prove smoothing effects for dispersive equations.

Multiplying the equation (1.1) by $2u$ and integrating in the x-variable yields, after integrating by parts,

$$\frac{d}{dt}\int_{\mathbb{R}} u^2(x,t)\,dx = -2\int_{\mathbb{R}} (\partial_x u)^2(x,t)\,dx. \tag{4.1}$$

Integrating in the time interval $[0,T]$ and using the fundamental theorem of calculus

$$\|u(T)\|_2^2 - \|u_0\|_2^2 = -2\int_0^T \|\partial_x u(t)\|_2^2\,dt.$$

By Theorem A, the left-hand side is finite if $u_0 \in L^2(\mathbb{R})$. Hence the right-hand side is finite. But

$$\int_0^T \|\partial_x u(t)\|_2^2\,dt < \infty \tag{4.2}$$

implies we can choose $0 < t^* \ll 1$ as small as desired so that

$$\|\partial_x u(t^*)\|_2 < \infty.$$

(It need not be true that $\|\partial_x u_0\|_2 < \infty$.) Now one can apply local well posedness in $H^1(\mathbb{R})$ beginning at time t^*, proving that the solution persists in this space for all $t \in [t^*, T]$. As t^* is arbitrary, we conclude

$$u(\cdot,t) \in H^1(\mathbb{R}) \quad \text{for all} \quad t > 0. \tag{4.3}$$

This demonstrates the gain of a single derivative in $L^2(\mathbb{R})$ for positive times, which isn't quite Theorem 1. The key insight required is to realize that differentiating equation (1.1) with respect to x produces

$$\partial_t(\partial_x u) = \partial_x^2(\partial_x u), \tag{4.4}$$

and so $\partial_x u$ also solves the heat equation! Therefore we may repeat the preceding argument to conclude

$$u(\cdot,t) \in H^2(\mathbb{R}) \quad \text{for all} \quad t > 0. \tag{4.5}$$

By induction, the parabolic smoothing result follows.

References

MR759907

[1] T. Kato, *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*, Studies in applied mathematics, Adv. Math. Suppl. Stud., vol. 8, Academic Press, New York, 1983, pp. 93–128. MR759907 (86f:35160)

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