Applications of Bessel's Functions

Casey Donlan and David Vartanian

†University of California Santa Barbara

Overview

One type of special function that is often encountered when solving physical problems is known as the Bessel function. They are solutions to a famous linear second order differential equation known as Bessel's equation. Bessel's equation is as follows:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0 \tag{1}$$

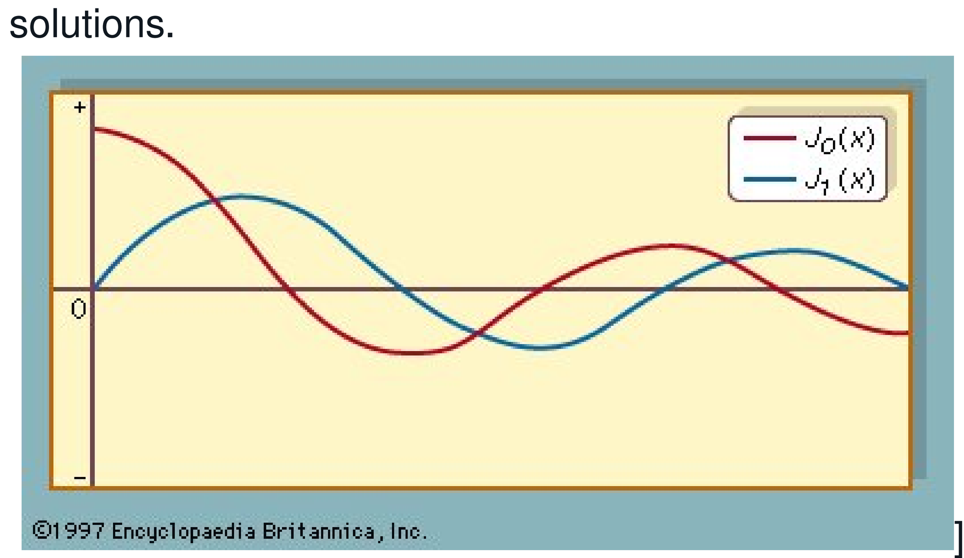
Here the number ν is called the order of the Bessel equation. The solutions to this equation are in the form of an infinite series and these solutions are known as Bessel's function of the first kind.

$$J_n(x) = x^n \sum_{m=1}^{\infty} 2^{-n} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!} \tag{2}$$

Bessel's functions are often encountered in situations involving cylindrical symmetry. As a result, there is a specific class of special functions known as spherical Bessel Functions.

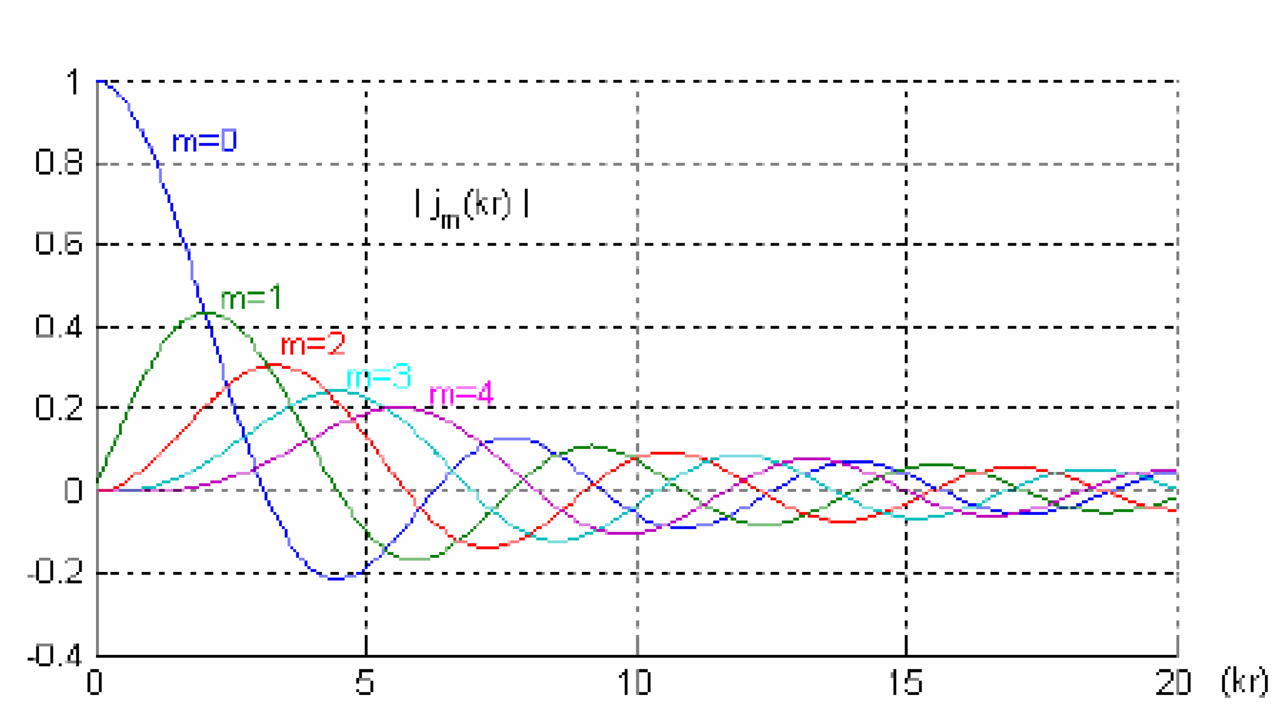
History

Bessel's Functions were derived around 1817 by German astronomer Freidrich Wilhelm Bessel. Bessel was trying to find a solution to one of Kepler's equations of planetary motion. Before Bessel, particular functions of the set equations had been created the Swiss mathematicians Daniel Bernoulli. Bernoulli studied the oscillations of a chain that was suspended by one end, and then another famous Swiss mathematician Leonhard Euler analyzed the vibrations of the stretched chain. When Bessel first published his findings, other scientists realized how powerful and useful these functions can be when trying to find a mathematical description for different physical phenomena. Furthermore, Lord Rayleigh was able to show that Bessel's functions arise in the solution of Laplace's equation when Laplace's equations are formulated in cylindrical coordinates. Since Bessel's functions are just solutions to Bessel's equation, we can graph these solutions.



] The graph of $J_0(x)$ looks just like a damped cosine curve, while $J_1(x)$ looks like a damped sine curve. Below is an image showing the solutions to the spherical Bessel's functions.

Fig. 1: Solutions to Bessel's equation.



Applications

As stated before, Bessel's Functions are very useful when it comes to mathematically modeling a variety of physical phenomena, especially when said phenomena is cylindrically or spherically symmetric. Some famous uses of Bessel's functions include the modeling of the flow of heat or electricity in a solid cylinder, the propagation of electromagnetic waves along wires, the diffraction of light, the motion of fluids, angular resolution, probability density function and the deformations of elastic bodies. The reason why Bessel's functions are often used in these situations is because Bessel's equation arises when finding separable solutions to Laplace's equations and the Helmholtz equation in cylindrical or spherical coordinates. Because of this, problems involving wave propagation and static potential heavily rely on Bessel's functions. When solving problems in cylindrical coordinate systems, Bessel functions of integer order $\alpha = n$ are used, but for spherical problems, half integer orders are used, $\alpha = (n + \frac{1}{2})$ One of the most important uses of Bessel's functions is when applying it to solve the Schrodinger's equation in a cylindrical well. If we consider a particle of mass m placed into a two dimensional potential well, where the potential is zero inside of the radius of the disk and infinite outside, then you can represent the system using polar coordinates. When solving the Schrodinger's equation the solutions are the spherical Bessel functions.

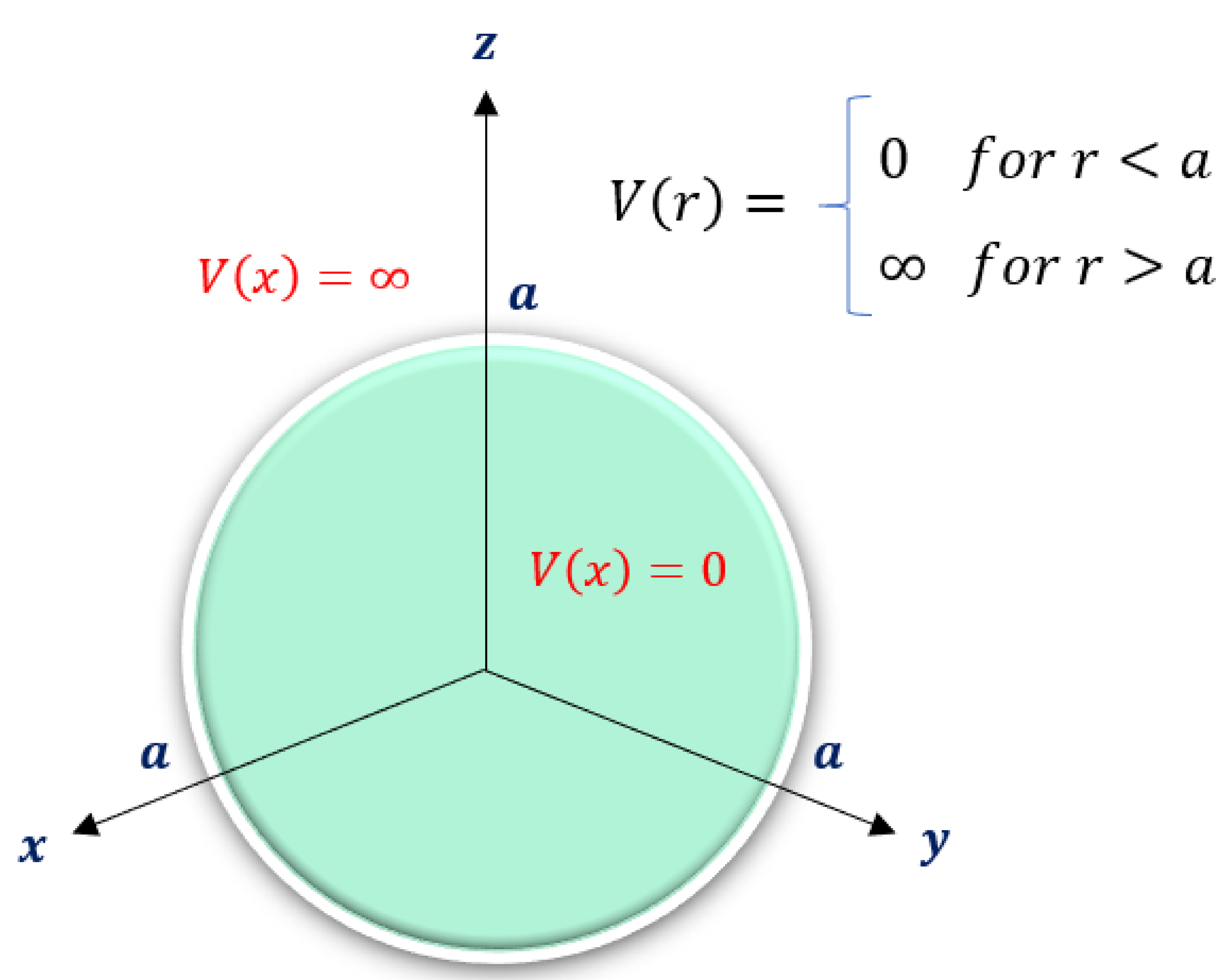
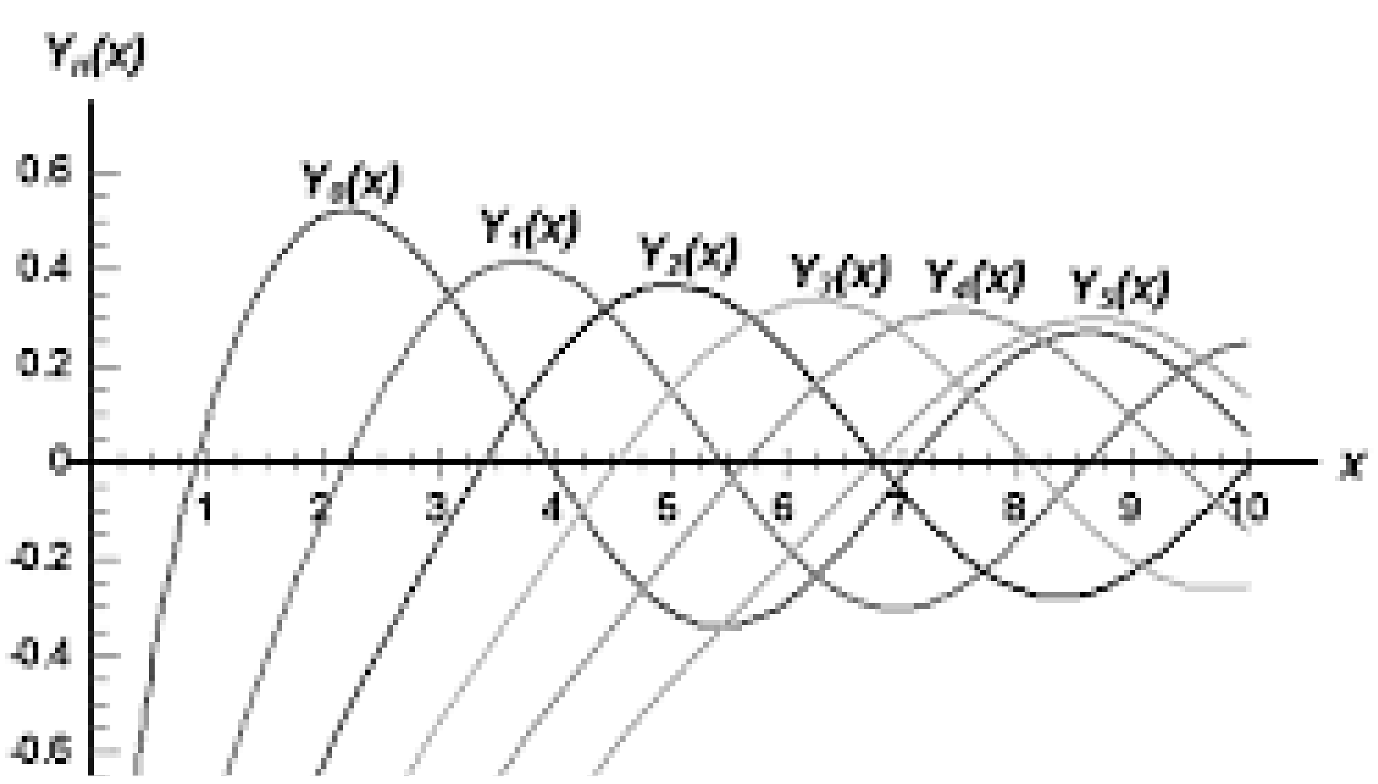


Fig. 3: Big fancy graphic.

Here is an image of the cylindrical well, and below is an image the solutions.



Examples

Bessel Equation of Order One-Half

$L[y] = x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$ where v=$\frac{1}{2}$

We let y=ϕ(r,x)=a$_0 x^r + \sum_{n=0}^{\infty} a_n x^{r+n}$

We then take the derivative of y to substitute into our equation

L$[\phi](r,x) = \sum_{n=0}^{\infty}[(r+n)(r+n-1) + (r+n) - \frac{1}{4}]a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2}$

After simplifying, we get $(r^2 - \frac{1}{4})^2 a_0 x^r + [(r+1)^2 - \frac{1}{4}]a_1 x^{r+1} + \sum_{n=2}^{\infty}[[(r+n)^2 - \frac{1}{4}]a_n + a_{n-2}]x^{r+n} = 0$

*Note that we changed the index of the summation in order to get the x^{r+n} term. This is needed to find the recurrence relation equation. The next step is to find the roots of the **indicial equation**. This is the equation we would typically use to find the $y = x^r$ solutions for a Euler Equation. These roots $(r_1$ and $r_2)$ are called the **exponents at singularity**. They determine the qualitative nature of the solution. In this equation, the exponents at singularity can be found by setting the x^r terms to zero. This gives us $r_1 = \frac{1}{2}$ and $r_2 = \frac{-1}{2}$

The **recurrence relation equation** uses the roots to generate patterns that tell us what the general solution of the equation will be. This equation is first found by setting the x^{r+n} term equal to zero.

That is, $[(r+n)^2 - \frac{1}{4}]a_n = -a_{n-2}$

Using the first root $(r_1 = \frac{1}{2})$, our recurrence relation is $a_n = \frac{-a_{n-2}}{n(n+1)}$

Let us set $n = 2m$ because the odd terms of the equation will always go to zero.

Letting, $n = 2m$, we have $a_{2m} = \frac{-a_{2m-2}}{2m(2m+1)}$ for $m = 1, 2, 3, ...$

Case 1: $m = 1$

$a_2 = \frac{-a_0}{3!}$

Case 2: $m = 2$

a$_4 = \frac{a_2}{20}$

Plug our a_2 term into the equation

a$_4 = \frac{a_0}{5!}$

We repeat this process so that each term has an a_0 in the equation. After evaluating this recurrence relation, we find that there is a repeating pattern of $a_{2m} = \frac{(-1)^m a_0}{(2m+1)!}$

The general formula for the first term looks like $y_1(x) = x^{r_1}[a_0 + \sum_{m=1}^{\infty} a_{2m} x^{2m}]$

Hence, our **first solution**, given that $a_0 = 1$, is

y$_1(x) = x^{1/2} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}$

You may recognize this solution. This is the Taylor series for sin x. Therefore, one solution of the One-Half Bessel Equation is $x^{\frac{-1}{2}} sin(x)$

We use J to denote our first solution, and the given formula for J$_{1/2} = (2/\pi)^{1/2} y_1$

Hence, $J_{1/2}(x) = (\frac{2}{\pi x})^{1/2} sin(x)$ is our first solution

Now we will use the second root $(r_2 = \frac{-1}{2})$ to find our second solution.

We go through the same process of recurrence relations, but we choose a set of even-numbered coefficients corresponding to a_0 and odd-numbered coefficients for a_1

The **second solution** ends up being

y$_2(x) = x^{-1/2}[a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}]$

This simplifies to $J_{-1/2}(x) = (\frac{2}{\pi x})^{1/2} cos(x)$

Our final, general solution of this Order One-Half Bessel Function ends up being

$$y = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$$

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