Abstract

The congruent number problem is a classical problem in Diophantine geometry that asks questions on the area of rational right triangles. The conditions for a rational number $n$ to be the area of a right triangle with rational side lengths was partially resolved by Tunnell in 1983. In this poster, we provide an overview of Tunnell’s theorem as presented in [1] as well as an implementation of an algorithm to determine whether a number $n$ is a so-called congruent number. Finally, we connect the theorem to the modern theory of elliptic curves.

The Algorithm

Our algorithm shown below calculates Tunnell’s Theorem by counting the number of solutions to the given equations given a certain number $n$, outputting whether $n$ is congruent or if the results are inconclusive. We were able to increase the efficiency of the algorithm by finding key observations that greatly limited the amount of numbers we need to check to find the number of solutions. Here were our observations:

1. Since $x$, $y$, and $z$ are all squared, every term in the left-hand side of the equation will always be positive. This tells us two things:
   (a) Each term in the sum on the left has an upper bound:
      i. For $f(n)$ and $g(n)$, each term cannot exceed $n$;
      ii. For $h(n)$ and $k(n)$, each term cannot exceed $n/2$.
   A. In particular, for example in $k(n)$, for the term $32z^2$, we deduce that all solutions must have $-\sqrt{n}/8 \leq z \leq \sqrt{n}/8$.
   (b) For every solution that contains a positive $x$, $y$, or $z$ in the solution, there exists a similar solution with a negative $x$, $y$, or $z$. Thus we don’t have to iterate through the possible negative solutions because they can all be accounted for through their positive counterparts.

2. Since the square of a number preserves its parity, $n$ must always be odd.
   (a) For odd $n$, we use $f(n)$ and $g(n)$.
      i. For $f(n)$, for all $y, z \in \mathbb{Z}$, $2y^2 + 8z^2$ is even. Thus $x$ must be odd in order for $x^2 + 2y^2 + 8z^2$ to be odd.
      ii. Similarly for $g(n)$, for all $y, z \in \mathbb{Z}$, $2y^2 + 32z^2$ is even so $n$ must be odd.
   (b) For even $n$, we use $h(n)$ and $k(n)$.
      i. For all even squarefree integers $n$, $n/2$ is necessarily odd. If it were even, then $4 - 2^n$ would imply that $n$ was not squarefree.
      ii. Similar logic can be applied as above to show that $x$ must be odd when considering $h(n)$ and $k(n)$.

Observation 1b is incredibly useful, as it halves the amount of numbers we need to iterate through to find solutions, however the only problem that arises is when $x$, $y$, and/or $z$ equals 0. Since 0 is neither positive nor negative, then the number of 0’s in a given solution affects the number of combinations of positive/negative solutions.

- For $x$ in range(1, int(math.sqrt(n) + 1), 2), $2y$ must be odd
- For $y$ in range(0, int(math.sqrt(n/2) + 1)): if $(x^2 + 2y^2 + 8z^2) == 0$:
  count = count + 1
  for $z$ in range(0, int(math.sqrt(n/2) + 1)):
    if $(x^2 + 2y^2 + 32z^2) == 0$:
      count = count + 1
  return count

Since Tunnell’s Theorem only works for $n$ squarefree, we also added an isSquareFree() function which returns whether or not a given input is squarefree. It does this in a brute-force way by checking if any square integer divides the input. We also verified our code for Tunnell’s theorem against the OEIS list of congruent numbers (sequence A003273) given in [2].

Problem Formulation

A triangle is called rational if all of its sides are rational. A positive rational number $n$ is called a congruent number if there exists a rational right triangle whose area is $n$. Equivalently, $n$ is a congruent number if there are $a, b, c \in \mathbb{Q}_{\geq 0}$ such that $a^2 + b^2 = c^2$ and $(1/2)ab = n$. The congruent number problem asks which positive rational numbers are congruent numbers. One can show that the question reduces to finding which squarefree integers are congruent numbers.

The congruent number problem is a classical problem in Diophantine geometry that intimately relates with the Birch and Swinnerton-Dyer conjecture. The Birch and Swinnerton-Dyer conjecture, one of the Millennium Prize problems, is an incredibly important conjecture in the field of arithmetic geometry, and even in mathematics as a whole. It relates elliptic curve data to the associated L-function. The details are not important for this poster. This connection to Elliptic curves comes from [3], in which Tunnell proves his theorem using the classical Jacobi theta function $g$ as well as the classical theta functions $\theta_2 = \sum_{n \in \mathbb{Z}} q^{n^2}$ and $\theta_3 = \sum_{n \in \mathbb{Z}} q^{n^4}$.

Relating these to known results about L-functions and elliptic curves, Tunnell found this characterization of congruent numbers.

In particular, the so-called congruent elliptic curve $y^2 = x^3 - n^2x$ is relevant because there is a bijection between rational right triangles $(a, b, c)$ with area $n$ and rational points $(x, y)$ on the elliptic curve $y^2 = x^3 - n^2x$. Thus we can use any tools for finding rational points on elliptic curves to identify congruent numbers.

Remarks

The congruent number problem, intimately related with the Birch and Swinnerton-Dyer conjecture, is nearly fully resolved, and indeed fully resolved provided the weak BSD is true for $y^2 = x^3 - n^2x$. Besides Tunnell’s theorem, another known partial classification of congruent numbers is that for any prime $p$, if $p \equiv 3 \pmod{8}$ then $2p$ is congruent, if $p \equiv 5 \pmod{8}$ then $p$ is congruent, and if $p \equiv 7 \pmod{8}$ then both $p$ and $2p$ are congruent.

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References


The Congruent Elliptic Curve $y^2 = x^3 - n^2x$