Connection between the gamma function and the Hurwitz zeta function

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Representations of the Gamma function

In the course of this poster, we will understand the gamma function to have real inputs. We can then use the following definition of the gamma function:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$

We will also be using the Weierstrass definition of the gamma function given by [2,p92]

$$\Gamma(x) = \frac{e^{-\gamma x}}{x}\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)^{-1}e^{x/n}$$

where $\gamma \approx 0.577216$ is the Euler-Mascheroni constant with the following construction [2,p19]:

$$\gamma = \lim_{n\to\infty}\left(-\log n + \sum_{k=1}^{n}\frac{1}{k}\right)$$

Motivation

Historically, the gamma function arouse from questions pertaining to the extension of the factorial function. From $n! = n(n-1)!$ where n is understood to be a natural number, we desire f with the recursion $f(x+1) = xf(x)$ with $f(1) = 1$. It is easily verifiable using the integral definition and integration by parts that $\Gamma(x)$ satisfies this for any $x \in \mathbb{R}$ except non-positive integers.

Uniqueness

Because $\Gamma(n+1) = n!$, it is conceivable that multiple continuous functions connecting the isolated factorial points could be drawn such that the recursion and $f(1) = 1$ are still satisfied. However, if we add a third condition of $\log(f(x))$ being convex, the Bohr-Mollerup theorem [1] says $f = \Gamma$ is now unique.

Graph of Gamma in the first quadrant

Below is the graph of $\Gamma(x)$ in the first quadrant accompanied by the first three tangent lines at natural number arguments of $\Gamma(x)$. Use this graph as reference in the next section.

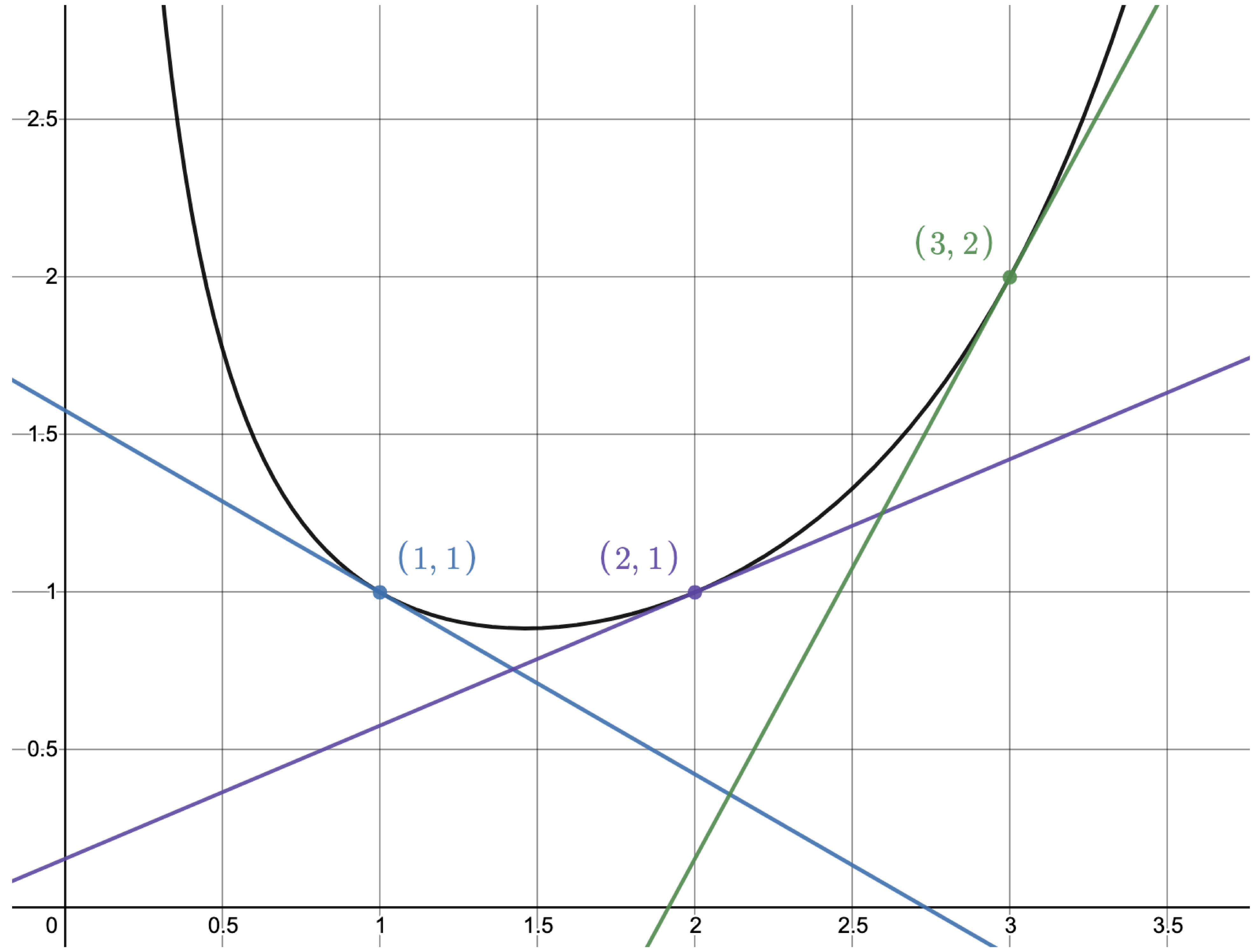

Figure 1:a graph of $\Gamma(x)$ and three of its tangent lines

Equations for these tangent lines

The following equations for these tangent lines are, in ascending order according to their point of tangency are

$$y_1 = -\gamma x + 1.5772... \quad \text{(blue line)}$$

$$y_2 = (1-\gamma)x + 0.1542... \quad \text{(purple line)}$$

$$y_3 = (3-2\gamma)x - 3.5364... \quad \text{(green line)}$$

Derivatives of Gamma

We will now turn our attention to derivatives of the Gamma function which will ultimately lead to a relationship to the Hurwitz zeta function. The polygamma function of order m, $\psi^{(m)}(x)$ is defined to be the $m+1$ derivative of $\log\Gamma(x)$ with respect to x. The digamma function, ψ is the polygamma function of order 0 which we will elaborate on next.

Digamma function

The digamma function, ψ is defined to be the logarithmic derivative of Γ, ie

$$D_x\log(\Gamma) = \frac{D_x\Gamma}{\Gamma} = \psi$$

From the Weierstrass definition of the Gamma function given before, we can rewrite this as follows

$$\begin{aligned}\psi(x) &= D_x\log\left(\frac{e^{-\gamma x}}{x}\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)^{-1}e^{x/n}\right)\\ &= D_x\left(-\gamma x - \log x + \sum_{n=1}^{\infty}\left(\frac{x}{n} - \log(1+\frac{x}{n})\right)\right)\\ &= -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{x+n}\right)\end{aligned}$$

Notice that for $x \in \mathbb{N}$ we have that the infinite series is telescopic and equal to the sum of the first x harmonic numbers, denoted H_x and $H_0 = 0$. So for $x \in \mathbb{N}$ we have the slope of the tangent line at $x = a$, and seen in Figure 1, will be

$$\Gamma(a)\psi(a) = \Gamma(a)(-\gamma - \frac{1}{a} + H_a) = \Gamma(a)(-\gamma + H_{a-1})$$

Hurwitz Zeta function

Below are the definitions for the Riemann Zeta function and Hurwitz Zeta function, respectively

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}, \quad \zeta(s,k) = \sum_{n=0}^{\infty}\frac{1}{(n+k)^s}$$

for $s > 1, k > 0$. Taking the derivative of ψ we arrive with the following equation

$$D_x\psi(x) = \frac{1}{x^2} + \sum_{n=1}^{\infty}\frac{1}{(x+n)^2} = \sum_{n=0}^{\infty}\frac{1}{(x+n)^2} = \zeta(2,x)$$

which is strictly positive for any x, verifying Γ is logarithmically convex.

References

[1] Bohr,H., Mollerup, J., (1922). Lærebog i Kompleks Analyse vol. III, Copenhagen.

[2] M.R.Murty,Problems in Analytic Number Theory, Springer Graduate Texts in Mathematics vol206, 2008, second ed.