GENERALIZED DIVISOR FUNCTIONS IN ARITHMETIC PROGRESSIONS: II

VARIANCE OF THE TERNARY DIVISOR FUNCTION IN ARITHMETIC PROGRESSIONS

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ABSTRACT. We give an asymptotic equality for the average of the variance for the ternary divisor function in arithmetic progressions. This estimate refines a recent conjecture about asymptotics of this variance. This result is also closely related to the problem of moments of Dirichlet L-function.

CONTENTS

1. INTRODUCTION

One form of the celebrated Bombieri-Vinogradov Theorem [3] [18] (1965) asserts that

$$\sum_{1\le q\le N^{1/2}(\log N)^{-B}} \max_{y\le N} \max_{(a,q)=1} \left| \sum_{\substack{1\le n\le y\\ n\equiv a(q)}} \Lambda(n) - \frac{y}{\varphi(y)} \right| \ll N(\log N)^{-A} \tag{1.1}$$

where $\Lambda(n)$ is the von Mangoldt function and $B = 4A + 40$ with $A > 0$ arbitrary. Analogues of (1.1) has been found for all $\tau_k(n)$ [15] and $\tau_2(n)^2$ [13, Lemma 8], where $\tau_k(n)$ is the k-fold divisor function: $\sum_{n=1}^{\infty} \tau_k(n)n^{-s} = \zeta^k(s)$.

Around the same time, Barban [1] [2] (1963-1964), Davenport-Halberstam [6] (1966), and Gallagher [7] (1967) found the following related inequality in which the absolute value is

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being squared:

$$\sum_{1\le q\le N(\log N)^{-B}} \sum_{\substack{1\le a\le q\\(a,q)=1}} \left| \sum_{\substack{1\le n\le N\\ n\equiv a(q)}} \Lambda(n) - \frac{N}{\varphi(q)} \right|^2 \ll N^2(\log N)^{-A}, \tag{1.2}$$

giving a much wider range for q. In fact, Davenport and Halberstam proved a slightly stronger result than Barban's, while Gallagher gave a simplified elegant proof. For this reason, this type of inequalities are often referred to as Barban-Davenport-Halberstram type inequalities.

Barban-Davenport-Halberstram type inequalities have many applications in number theory. For instance, a version of this inequality (with $\Lambda(n)$ replaced by related convolutions over primes) was skillfully used by Zhang [19, Lemma 10] (2014) in his spectacular work on bounded gaps between primes.

Shortly after, in 1970 Montgomery [12] succeeded in replacing the inequality in (1.2) by an asymptotic equality. One of his results is

$$\sum_{1\le q\le Q} \sum_{\substack{1\le a\le q\\(a,q)=1}} \left| \sum_{\substack{1\le n\le N\\ n\equiv a(q)}} \Lambda(n) - \frac{N}{\varphi(q)} \right|^2 = QN\log N + O(QN\log(2N/Q)) + O\left(N^2(\log N)^{-A}\right) \tag{1.3}$$

for $Q \le N$ and $A > 0$ arbitrary. A few years latter, Hooley [9] (1975), by introducing new ideas in treatment of the off-diagonal terms specific to primes, sharpened the right side of (1.3) to

$$QN\log N + O(QN) + O(N^2(\log N)^{-A}))$$

with $\Lambda(n)$ replaced by the Chebyshev function $\theta(n)$.

Motohashi [14] (1973), by using an approach similar to Montgomery, elaborately established a more precise asymptotic with lower order and power saving error terms for the divisor function $\tau(n)$. Recently, by function field analogues, Rodgers and Soundararajan [16] (2018) were led to the following conjecture for the variance of the k-fold divisor function τ_k over the integers.

Conjecture 1.1. *For $X, d \to \infty$ such that $\log X/\log d \to c \in (0, k)$, we have*

$$\sum_{\substack{a=1\\(a,d)=1}}^{d} \left| \sum_{\substack{1\le n\le X\\ n\equiv a(\mathit{mod}\ d)}} \tau_k(n) - \frac{1}{\varphi(d)} \sum_{\substack{1\le n\le X\\(n,d)=1}} \tau_k(n) \right|^2 \sim a_k(d)\gamma_k(c)X(\log d)^{k^2-1},$$

where $a_k(d)$ is the arithmetic constant

$$a_k(d) = \lim_{s\to 1^+} (s-1)^{k^2} \sum_{\substack{n=1\\(n,d)=1}}^{\infty} \frac{\tau_k(n)^2}{n^s},$$

and $\gamma_k(c)$ is a piecewise polynomial of degree k^2-1 defined by

$$\gamma_k(c) = \frac{1}{k!G(k+1)^2}\int_{[0,1]^k} \delta_c(w_1+\cdots w_k)\Delta(w)^2 d^k w,$$

where $\delta_c(x) = \delta(x-c)$ is a Dirac delta function centered at c, $\Delta(w) = \prod_{i<j}(w_i - w_j)$ is a Vandermonde determinant, and G is the Barnes G-function, so that in particular $G(k+1) = (k-1)!(k-2)!\cdots 1!$.

This conjecture is closely related to the problem of moments of Dirichlet L-functions [4] and correlations of divisor sums [5]. In the same paper [16], Rodgers and Soundararajan confirmed an averaged version of this conjecture in a restricted range over smooth cutoffs. Harper and Soundararajan [8] obtained a lower bound of the right order of magnitude for the average of this variance. By using the large sieve inequality, Nguyen [15, Theorem 1] obtained an upper bound of the same order of magnitude for this averaged variance.

In this paper, we replace the upper bound in [15, Theorem 1] by an asymptotic equality for the ternary divisor function $\tau_3(n)$ averaged over moduli up to the length of the sum. Our approach is based on Motohashi's treatment for the case of τ_2, with appropriate modifications; see Section 2.1 below for a discussion and possible extensions.

1.1. **Notations.** $\tau_k(n)$: the number of ways to write a natural number n as an ordered product of k positive integers.

$\tau(n) = \tau_2(n)$: the usual divisor function.

$\varphi(n)$: Euler's function, i.e., the number of reduced residue classes modulo n.

$\zeta(s)$: Riemann's zeta function with variable $s = \sigma + it$.

$\Gamma(s)$: Gamma function.

γ: Euler's constant $= 0.5722\ldots$.

$\gamma_0(\alpha)$: 0-th generalized Stieltjes constant

$$\gamma_0(\alpha) = \lim_{m\to\infty}\left(\sum_{k=0}^{m}\frac{1}{k+\alpha} - \log(m+\alpha)\right).$$

$e(x) = e^{2\pi i x}$.

$e_q(a) = e^{2\pi i \frac{a}{q}}$.

$c_q(b)$: Ramanujan's sum

$$c_q(b) = \sum_{\substack{1\le a\le q\\(a,q)=1}} e_q(ab).$$

(m,n): the greatest common divisor of m and n.

$[m,n]$: the least common divisor of m and n.

N: sufficiently large integer.

ε: arbitrary small positive constant, not necessarily the same in each occurrence.

$P_r(\log N)$: a polynomial of degree r in $\log N$, not necessarily the same in each occurrence.

Throughout the paper, all constants in O-terms or in Vinogradov's notation $\ll$ depends on ε at most.

1.2. **Acknowledgments.** I thank Soundararajan for pointing out the reference [8] to me at the AIM FRG Seminar, which then motivated me to work on this problem, and Brad Rodgers and Jeff Stopple for their interests in this project. This work was done while I was visiting the American Institute of Mathematics, virtually, which I am very grateful for their hospitality.

2. Statement of result

Our main result is the following

THEOREM 2.1. *We have the following asymptotic equality, with effectively computable numerical constants*

$$\mathfrak{S}_j,\ (0 \leq j \leq 8),$$

$$\sum_{1\leq \ell \leq N} \sum_{1\leq b\leq \ell} \left| \sum_{\substack{1\leq n\leq N \\ n\equiv b (mod\ \ell)}} \tau_3(n) - NP_2(\log N) \right|^2 = N^2 \sum_{j=0}^{8} \mathfrak{S}_{8-j} \log^{8-j} N + O\left(N^{599/300}\right), \tag{2.1}$$

where

$$P_2(\log N) = \operatorname*{Res}_{s=1} \left\{ \sum_{n\equiv b (mod\ \ell)} \frac{\tau_3(n)}{n^s} \frac{N^{s-1}}{s} \right\} = \frac{1}{2}\tilde{A} \log^2 N - (\tilde{A} - \tilde{B}) \log N + (\tilde{A} - \tilde{B} + \tilde{C}),$$

with

$$\tilde{A} = \tilde{A}(\ell, b) = \ell^{-1} \sum_{q|\ell} q^{-3} c_q(b) \sum_{\alpha,\beta,\gamma=1}^{q} e_q(a\alpha\beta\gamma),$$

$$\tilde{B} = \tilde{B}(\ell, b) = \ell^{-1} \sum_{q|\ell} q^{-3} c_q(b) \sum_{\alpha,\beta,\gamma=1}^{q} e_q(a\alpha\beta\gamma)(3\gamma_0(\alpha/q) - 3\log q),$$

$$\tilde{C} = \tilde{C}(\ell, b) = \ell^{-1} \sum_{q|\ell} q^{-3} c_q(b) \sum_{\alpha,\beta,\gamma=1}^{q} e_q(a\alpha\beta\gamma)(3\gamma_0(\alpha/q)\gamma_0(\beta/q) - 9\gamma_0(\alpha/q)\log q + \frac{9}{2}\log^2 q),$$

γ is Euler's constant, $\gamma_0(\alpha)$ is the 0-th Stieltjes constant, and $c_q(b)$ is the Ramanujan sum.

The constants $\mathfrak{S}_j, 0 \leq j \leq 8$, have complicated expressions but can be explicitly determined from our proof. We give here the value of the leading constant $\mathfrak{S}_8$:

$$\mathfrak{S}_8 = \frac{1}{8!} \prod_p \left(1 - 9p^{-2} + 16p^{-3} - 9p^{-4} + p^{-6}\right) = 1.223\cdots \times 10^{-6}.$$

We note that it is possible to obtain asymptotics for (2.1) with ℓ averaged over the range $1 \leq \ell \leq L$ for $L < N$ and $L > N$. A phase transition in the coefficient of the leading asymptotic might begin to occur. It is also plausible to use this method, in conjunction with subconvexity bounds for $\zeta(s)$, to treat the variance of $\tau_4(n)$. On the generalized Riemann hypothesis, it might also be possible to treat all $\tau_k(n)$. We hope to return to these ideas in a future article.

2.1. **Outline of the proof.** We follow the approach of Motohashi [14] in his treatment of the divisor function $\tau(n)$, which in turn was based on Montgomery's adaptation [12] of a result of Lavrik [11] on twin primes on average.

To control the error term, we prove an analog of Lavrik's result for τ_3, using a simpler version of Vinogradov's method of trigonometric sums, as in Motohashi. The standard convexity bound for $\zeta(s)$ in the critical strip suffices for our purpose. We remark here that our analogue of Lavrik's result can be seen as an average result concerning the mean square error of the following modified additive divisor sum

$$\sum_{1\le n\le N-h} \tau_3(n)\tau_3(n+h)$$

of length $N-h$ for averaged over h up to $h \le N-1$. This idea might also have application to the sixth power moment of $\zeta(s)$, which we plan to revisit in the near future.

To evaluate the main term, we proceed slightly different from Motohashi due to some complications involving an exponential sum in three variables. We show that the resulting sum can be evaluated, on average, thanks to the orthogonality property of the Ramanujan's sum.

3. Preparatory lemmas

For $\sigma > 1$ and $(a, q) = 1$, let

$$E\left(s;\frac{a}{q}\right) = E_3\left(s;\frac{a}{q}\right) = \sum_{n=1}^{\infty} \tau_3(n)e_q(an)n^{-s}. \tag{3.1}$$

The case for the usual divisor function $\tau(n)$ was considered by Hecke and Estermann (1930). Smith (1982) extended the result to all τ_k. We specialize to a special case his results.

LEMMA 3.1. *[17, Theorem 1, pg. 258] The function $E_3(s; a/q)$ has a meromorphic continuation to the whole complex plane where it is everywhere holomorphic except for a pole of order 3 at $s = 1$. Moreover, $E(s; a/q)$ satisfies the functional equation*

$$E(s;a/q) = \left(\frac{q}{\pi}\right)^{-\frac{3}{2}(2s-1)} \frac{\Gamma^3\left(\frac{1-s}{2}\right)}{\Gamma^3(\frac{s}{2})} E^+(1-s;a/q) + i\left(\frac{q}{\pi}\right)^{\frac{3}{2}(2s-1)} \frac{\Gamma^3\left(\frac{2-s}{2}\right)}{\Gamma^3\left(\frac{1+s}{2}\right)} E^-(1-s;a/q), \tag{3.2}$$

where

$$E^{\pm}(s;a/q) = \sum_{m_1,m_2,m_3\ge 1} G^{\pm}(m_1,m_2,m_3;a/q)(m_1m_2m_3)^{-s}, \quad (\sigma > 1),$$

$$G^{\pm}(m_1,m_2,m_3;a/q) = \frac{1}{2q^{3/2}}\{G(m_1,m_2,m_3;a/q) \pm G(m_1,m_2,m_3;-a/q)\},$$

and

$$G(m_1,m_2,m_3;a/q) = \sum_{x_1,x_2,x_3(q)} e_q(am_1m_2m_3 + m_1x_1 + m_2x_2 + m_3x_3).$$

We rewrite the functional equation (3.2) as follows (c.f. Ivic [10]). Let

$$A^{\pm}(n, a/q) = \sum_{n_1n_2n_3=n} \sum_{x_1,x_2,x_3=1}^{q} \frac{1}{2}\big(e_q(ax_1x_2x_3 + n_1x_1 + n_2x_2 + n_3x_3) \pm e_q(-ax_1x_2x_3 + n_1x_1 + n_2x_2 + n_3x_3)\big).$$

We have that

$$|A^{\pm}(n, a/q)| \leq q^3\tau_3(n).$$

Then from Lemma 3.1 we obtain the following form of the functional equation.

LEMMA 3.2. *[10, Lemma 2, pg. 1007] For $\sigma < 0$ and $(a, q) = 1$, we have*

(3.3)
$$E(s; a/q) = \left(\frac{q}{\pi}\right)^{-\frac{3}{2}(2s-1)} \left\{ \frac{\Gamma^3\left(\frac{1-s}{2}\right)}{\Gamma^3\left(\frac{s}{2}\right)} \sum_{n=1}^{\infty} A^+(n, a/q)n^{s-1} + i\frac{\Gamma^3\left(\frac{2-s}{2}\right)}{\Gamma^3\left(\frac{1+s}{2}\right)} \sum_{n=1}^{\infty} A^-(n, a/q)n^{s-1} \right\},$$

where the two series on the right-side are absolutely convergent.

We also need the Laurent expansion of $E(s; a/q)$ at $s = 1$ for residue calculations.

LEMMA 3.3. *For $(a, q) = 1$, we have*

$$E(s; a/q) = \frac{1}{q}\left(\frac{A}{(s-1)^3} + \frac{B}{(s-1)^2} + \frac{C}{s-1}\right) + \sum_{n=0}^{\infty} c_n(a, q)(s-1)^n, \tag{3.4}$$

where

$$A = A(q) = q^{-2} \sum_{\alpha,\beta,\gamma=1}^{q} e_q(a\alpha\beta\gamma),$$

$$B = B(q) = q^{-2} \sum_{\alpha,\beta,\gamma=1}^{q} e_q(a\alpha\beta\gamma)(3\gamma_0(\alpha/q) - 3\log q),$$

$$C = C(q) = q^{-2} \sum_{\alpha,\beta,\gamma=1}^{q} e_q(a\alpha\beta\gamma)(3\gamma_0(\alpha/q)\gamma_0(\beta/q) - 9\gamma_0(\alpha/q)\log q + \frac{9}{2}\log^2 q),$$

with

$$\gamma_0(\alpha) = \lim_{m\to\infty}\left(\sum_{k=0}^{m} \frac{1}{k+\alpha} - \log(m+\alpha)\right).$$

The coefficients A, B, C are independent of a and satisfy

$$\begin{aligned} A(q) &\ll \tau(q)\log^2 q, \\ B(q) &\ll \tau(q)\log^3 q, \\ C(q) &\ll \tau(q)\log^4 q \end{aligned} \tag{3.5}$$

uniformly in a.

LEMMA 3.4. *For $n \geq 1$ and $(a, q) = 1$, we have*

$$\operatorname*{Res}_{s=1} E\left(s; \frac{a}{q}\right)\frac{n^s}{s} = q^{-1}n\left(\frac{A}{2}\log^2 n - (A - B)\log n + (A - B + C)\right), \tag{3.6}$$

where A, B, C are given in Lemma 3.3.

Proof. We have, by (3.4),

$$\begin{aligned}\operatorname*{Res}_{s=1} E\left(s;\frac{a}{q}\right)\frac{n^s}{s} &= \frac{1}{2}\lim_{s\to 1}\frac{d^2}{ds^2}\left((s-1)^3 E\left(s;\frac{a}{q}\right)\frac{n^s}{s}\right)\\ &= \frac{1}{2q}\lim_{s\to 1}\frac{d^2}{ds^2}\left((A+B(s-1)+C(s-1)^2+O((s-1)^3))\frac{n^s}{s}\right)\\ &= q^{-1}n\left(\frac{A}{2}\log^2 n-(A-B)\log n+(A-B+C)\right).\end{aligned}$$

□

LEMMA 3.5. *For $\sigma > 1$, let*

$$R(s;\ell,b)=\sum_{n\equiv b(mod\ \ell)}\tau_3(n)n^{-s}.$$

We have

$$\operatorname*{Res}_{s=1} R(s;\ell,b)\frac{N^s}{s}=N\left(\frac{\tilde{A}}{2}\log^2 N-(\tilde{A}-\tilde{B})\log N+(\tilde{A}-\tilde{B}+\tilde{C})\right),$$

where

$$\begin{aligned}\tilde{A}=\tilde{A}(\ell,b)&=\ell^{-1}\sum_{q|\ell}q^{-1}c_q(b)A(q),\\ \tilde{B}=\tilde{B}(\ell,b)&=\ell^{-1}\sum_{q|\ell}q^{-1}c_q(b)B(q),\\ \tilde{C}=\tilde{C}(\ell,b)&=\ell^{-1}\sum_{q|\ell}q^{-1}c_q(b)C(q),\end{aligned}$$

with $A(a), B(q), C(q)$ given in Lemma 3.3.

Proof. We can write $R(s;\ell,b)$ as

$$\begin{aligned}R(s;\ell,b)&=\frac{1}{\ell}\sum_{q|\ell}\sum_{\substack{1\le a\le q\\(a,q)=1}}e_q(-ab)E\left(s;\frac{a}{q}\right)\\ &=\frac{1}{\ell}\sum_{q|\ell}\frac{1}{q}c_q(b)\left(\frac{A(q)}{(s-1)^3}+\frac{B(q)}{(s-1)^2}+\frac{C(q)}{s-1}\right)+\sum_{n=0}^{\infty}\frac{1}{\ell}\sum_{q|\ell}\frac{1}{q}c_q(b)c_n(a,q)(s-1)^n\\ &=\frac{\tilde{A}(\ell,b)}{(s-1)^3}+\frac{\tilde{B}(\ell,b)}{(s-1)^2}+\frac{\tilde{C}(\ell,b)}{s-1}+\sum_{n=0}^{\infty}\frac{1}{\ell}\sum_{q|\ell}\frac{1}{q}c_q(b)c_n(a,q)(s-1)^n.\end{aligned}$$

The lemma follows as in the previous one. □

For $\alpha\in\mathbb{R}$, let

$$D(\alpha,N)=\sum_{1\le n\le N}\tau_3(n)e(\alpha n). \tag{3.7}$$

Using (3.1) we first estimate $D(\alpha,N)$ for $\alpha=a/q$ with $(a,q)=1$.

LEMMA 3.6. *For $(a,q)=1$, we have*

$$D\left(\frac{a}{q},n\right)=\frac{n}{q}\left(\frac{A}{2}\log^2 n-(A-B)\log n+(A-B+C)\right)+O\left\{(nq+q^2)^{3/5+\varepsilon}\right\},$$

with A,B,C given in Lemma 3.3.

Proof. We have

$$D\left(\frac{a}{q},n\right)=\operatorname*{Res}_{s=1}E\left(s;\frac{a}{q}\right)\frac{n^s}{s}+\operatorname*{Res}_{s=0}E\left(s;\frac{a}{q}\right)\frac{n^s}{s}+\frac{1}{2\pi i}\int_{-\delta-iT}^{-\delta+iT}E\left(s;\frac{a}{q}\right)\frac{n^s}{s}ds \\ +O\left\{\frac{n^{1+\varepsilon}}{T}+n^{\varepsilon}+\frac{1}{T}\int_{-\delta}^{1+\delta}\left|E\left(\sigma+iT;\frac{a}{q}\right)\right|n^{\sigma}d\sigma\right\}, \tag{3.8}$$

where $\delta=(\log(nq+1))^{-1}$ and T is to be determined latter. By expressing the residue as an integral around the origin,

$$\left|\operatorname*{Res}_{s=0}E\left(s;\frac{a}{q}\right)\frac{n^s}{s}\right|\ll(\log(qn+1))^3. \tag{3.9}$$

By the functional equation (3.3) and the convexity argument,

$$\left|E\left(\sigma+iT;\frac{a}{q}\right)\right|\ll(qT)^{\frac{3}{2}(1-\sigma)}(\log qT)^6$$

uniformly for $-\delta\le\sigma\le 1+\delta$. Hence we get

$$\left|\frac{1}{2\pi i}\int_{-\delta-iT}^{-\delta+iT}E\left(s;\frac{a}{q}\right)\frac{n^s}{s}ds\right|\ll(Tq)^{\frac{3}{2}}(\log qT)^7 \tag{3.10}$$

and

$$\frac{1}{T}\int_{-\delta}^{1+\delta}\left|E\left(\sigma+iT;\frac{a}{q}\right)\right|n^{\sigma}d\sigma\ll\frac{n}{T}(\log qT)^6\int_{-\delta}^{1+\delta}\left(\frac{Tq}{n^{2/3}}\right)^{\frac{3}{2}(1-\sigma)}d\sigma. \tag{3.11}$$

Taking

$$T=q^{-1}(nq+q^2)^{2/5}$$

it follows from (3.6), (3.8), (3.9), (3.10) and (3.11) that

$$D\left(\frac{a}{q},n\right)=\frac{n}{q}\left(\frac{A}{2}\log^2 n-(A-B)\log n+(A-B+C)\right)+O\left\{(nq+q^2)^{3/5+\varepsilon}\right\}.$$

□

LEMMA 3.7. *For $\alpha\in\mathbb{R}$, we have*

$$D(\alpha,N)=\frac{1}{q}\sum_{1\le n\le N}\left(\frac{A}{2}\log^2 n-(A-B)\log n+(A-B+C)\right)e\left(\left(\alpha-\frac{a}{q}\right)n\right) \\ +O\left\{(Nq+q^2)^{3/5+\varepsilon}\left(1+\left|\alpha-\frac{a}{q}\right|N\right)\right\}, \tag{3.12}$$

with A,B,C given in Lemma 3.3.

Proof. We have

$$D(\alpha, N) = \sum_{1\le n\le N} \{D(a/q, n) - D(a/q, n-1)\} \, e((\alpha - a/q)n).$$

This, together with Lemma 3.6 and partial summation, gives (3.12). ☐

Let

$$F\left(\alpha, \frac{a}{q}, N\right) = \frac{1}{q} \sum_{1\le n\le N} \left(\frac{A}{2}\log^2 n - (A-B)\log n + (A-B+C)\right) e\left(\left(\alpha - \frac{a}{q}\right) n\right) \tag{3.13}$$

and

$$G_\Delta(\alpha, N) = \sum_{1\le q\le \Delta} \sum_{\substack{a=1\\(a,q)=1}}^{q} \left| F\left(\alpha, \frac{a}{q}, N\right)\right|^2, \tag{3.14}$$

where Δ satisfies

$$4\Delta \le \Omega \tag{3.15}$$

and Δ is to be determined more precisely latter; see (4.7) below. By Lemma 3.7 and equation (3.13),

$$|D(\alpha, N) - F(\alpha, a/q, N)| \ll (Nq + q^2)^{3/5+\varepsilon} \left(1 + \left|\alpha - \frac{a}{q}\right| N\right). \tag{3.16}$$

Now, by (3.13) and (3.14),

$$G_\Delta(\alpha, N) = \sum_{|k|\le N-1} e(\alpha k) \left(\sum_{1\le q\le \Delta} \frac{1}{q^2} W_q(k, N) \sum_{\substack{a=1\\(a,q)=1}}^{q} e_q(-ak) \right), \tag{3.17}$$

where

$$\begin{aligned} W_q(k, N) = {} & \frac{1}{4}A^2 \sum_{1\le n\le N-|k|} \log^2 n \log^2(n+|k|) \\ & - \frac{1}{2}A(A-B) \sum_{1\le n\le N-|k|} \log n \log(n+|k|) \log n(n+|k|) \\ & + (A-B)^2 \sum_{1\le n\le N-|k|} \log n \log(n+|k|) \\ & - \frac{1}{2}A(A-B+C) \sum_{1\le n\le N-|k|} (\log^2 n + \log^2(n+|k|)) \\ & - (A-B)(A-B+C) \sum_{1\le n\le N-|k|} \log n(n+|k|) \\ & + (A-B+C)^2(N-|k|) \\ = {} & w_1(q)T_1(k, N) + \cdots + w_6(q)T_6(k, N), \end{aligned} \tag{3.18}$$

say. For the innermost sum in (3.17) we have

$$\sum_{\substack{a=1\\(a,q)=1}}^{q} e_q(-ak) = \mu\left(\frac{q}{(q,|k|)}\right)\frac{\varphi(q)}{\varphi\left(\frac{q}{(q,|k|)}\right)} = c_q(|k|).$$

Thus we write (3.17) as

$$G_\Delta(\alpha, N) = \sum_{|k|\le N-1}\left(\sum_{1\le q\le\Delta} q^{-2}c_q(|k|)W_q(k,N)\right)e(\alpha k) = \sum_{|k|\le N-1} S_\Delta(k,N)e(\alpha k), \tag{3.19}$$

say. Now, by (3.7), we have

$$|D(\alpha,N)|^2 = \sum_{|k|\le N-1} V(k,N)e(\alpha k),$$

where

$$V(k,N) = \sum_{1\le n\le N-|k|} \tau_3(n)\tau_3(n+|k|). \tag{3.20}$$

Thus,

$$|D(\alpha,N)|^2 - G_\Delta(\alpha,N) = \sum_{|k|\le N-1}(V(k,N)-S_\Delta(k,N))e(\alpha k).$$

and we obtain

LEMMA 3.8.

$$\sum_{|k|\le N-1}(V(k,N)-S_\Delta(k,N))^2 = \int_0^1 \left||D(\alpha,N)|^2 - G_\Delta(\alpha,N)\right|^2 d\alpha, \tag{3.21}$$

with $D(\alpha,N)$, $G_\Delta(\alpha,N)$, $V(k,N)$, and $S_\Delta(k,N)$ given by (3.7), (3.14), (3.20), and (3.19), respectively.

This integral will be estimated in Section 4 below.

LEMMA 3.9. *With*

$$\begin{aligned}
T_1(k,N) &= \sum_{1\le n\le N-|k|} \log^2 n \log^2(n+|k|),\\
T_2(k,N) &= \sum_{1\le n\le N-|k|} \log n \log(n+|k|)\log(n(n+|k|)),\\
T_3(k,N) &= \sum_{1\le n\le N-|k|} \log n \log(n+|k|),\\
T_4(k,N) &= \sum_{1\le n\le N-|k|} (\log^2 n + \log^2(n+|k|)),\\
T_5(k,N) &= \sum_{1\le n\le N-|k|} \log(n(n+|k|)).
\end{aligned} \tag{3.22}$$

given from (3.18)*, we have*

$$T_1(k,N) = (N-|k|)\log^2 N \log^2(N-|k|) + O(N\log^3 N),$$
$$T_2(k,N) = (N-|k|)(\log^2 N\log(N-|k|) + \log N\log^2(N-|k|)) + O(N\log^2 N),$$
$$T_3(k,N) = (N-|k|)\log N\log(N-|k|) + O(N\log N),$$
$$T_4(k,N) = (N-|k|)(\log^2 N + \log^2(N-|k|)) + O(N\log N),$$
$$T_5(k,N) = (N-|k|)(\log N + \log(N-|k|)) + O(N).$$

Proof. For $k>0$, by partial summation, we have

$$T_5(k,N) = (N-k)\log(N-k) + N\log N - k\log k - 2(N-k) + O(\log N).$$

Similarly, we obtain the other T_j's. □

LEMMA 3.10. *For positive integer δ and $q>1$, we have*

$$\sum_{1\le m\le X} c_q(\delta m) = O(q\delta^{-1/2}X^{1/2}\log^2 X).$$

Proof. Let us consider the function

$$f(s) = \sum_{m=1}^{\infty}\frac{c_q(\delta m)}{m^s}.$$

We have

$$f(s) = \sum_{m=1}^{\infty}\sum_{\substack{d|\delta m\\ dd'=q}}\mu(d')dm^{-s} = \delta^{-s}\sum_{m=1}^{\infty}m^{-s}\sum_{d|q}\mu\left(\frac{q}{d}\right)d^{1-s} = \delta^{-s}\zeta(s)\sum_{d|q}\mu\left(\frac{q}{d}\right)d^{1-s}.$$

Hence we have

$$\sum_{1\le m\le X} c_q(\delta m) = \operatorname*{Res}_{s=1}\left\{\delta^{-s}\zeta(s)\sum_{d|q}\mu\left(\frac{q}{d}\right)d^{1-s}\frac{X^s}{s}\right\} + \frac{1}{2\pi i}\int_{1/2-iT}^{1/2+iT}\delta^{-s}\zeta(s)\sum_{d|q}\mu\left(\frac{q}{d}\right)d^{1-s}\frac{X^s}{s}ds$$
$$+ O\left\{\frac{X^{1+\varepsilon}}{T} + X^{\varepsilon} + \frac{X^{\varepsilon}}{T}\int_{1/2}^{1}|\zeta(\sigma+iT)|\left|\sum_{d|q}\mu\left(\frac{q}{d}\right)d^{1-\sigma-iT}\right|X^{\sigma}d\sigma\right\}.$$

Since we have

$$\int_{-T}^{T}|\zeta(1/2+it)|\frac{dt}{|t|+1} \ll \log^2 T,$$

$$\left|\sum_{d|q}\mu\left(\frac{q}{d}\right)d^{1-s}\right| \le \sum_{d|q}d^{1/2} \ll q,$$

and

$$|\zeta(\sigma+iT)| \ll T^{\frac{1}{3}(1-\sigma)}\log^5 T;$$

taking $T=X$ we complete the proof. □

We will apply Perron's formula in the following form.

LEMMA 3.11. *Let $f(s)=\sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series converges absolutely for $\sigma>1$. Suppose $a_n=O(n^{\varepsilon})$ for any $\varepsilon>0$ and $f(s)=\zeta(s)^{\ell}F(s)$ for some natural number ℓ and some Dirichlet series $F(s)$ absolutely converges in $\Re(s)>1/2$. Then for X not an integer, we have*

$$\sum_{n\le X} a_n=\frac{F(1)}{(\ell-1)!}XP_{\ell-1}(\log X)+O_{\varepsilon}\left(X^{1-\frac{1}{\ell+2}}\right),$$

where $P_{\ell-1}(\log X)$ is the polynomial in $\log X$ of degree $\ell-1$ with leading coefficient 1 given explicitly by

$$P_{\ell-1}(\log X)=(\ell-1)!Res_{s=1}\zeta(s)^{\ell}F(s)\frac{X^{s-1}}{s}.$$

LEMMA 3.12. *We have*

$$\sum_{n\le X}\tau_3^2(n)=\frac{a_3}{8!}XP_8(\log X)+O\left(X^{10/11}\right),$$

where

$$a_3=\prod_p\left(1-9p^{-2}+16p^{-3}-9p^{-4}+p^{-6}\right)=0.04932\ldots$$

and $P_8(\log X)$ is a polynomial of degree 8 in $\log X$ and leading coefficient 1.

Proof. We have

$$\sum_{n=1}^{\infty}\tau_3^2(n)n^{-s}=\prod_p\left\{1+\sum_{\nu=1}^{\infty}\binom{\nu+2}{2}^2p^{-\nu s}\right\},$$

where both members of this equation are absolutely convergent if $\sigma>1$. Hence, if $\sigma>1$,

$$\begin{aligned}\{\zeta(s)\}^{-9}\left\{\sum_{n=1}^{\infty}\tau_3^2(n)n^{-s}\right\}&=\prod_p\left\{(1-p^{-s})^9(1+9p^{-s}+36p^{-2s}+\cdots)\right\}\\&=\prod_p\left\{1+a_2p^{-2s}+a_3p^{-3s}+\cdots\right\}=F(s),\end{aligned}$$

say, where

$$a_\nu=\sum_{r=0}^{\nu}(-1)^r\binom{9}{r}\binom{\nu-r+2}{2}^2.$$

We adpot the convention for the binomial coefficients that $\binom{n}{m}=0$ if $m>n$. The coefficient a_ν satisfies

$$|a_\nu|\le K\nu^2,$$

where K is independent of ν. Hence

$$\sum_{\nu=2}^{\infty}|a_\nu|p^{-\nu s}\le K'p^{-2s},$$

where K' is independent of p. Hence, if $\sigma > 1/2$, then $\sum_p p^{-2s}$ is absolutely convergent, and thus is also

$$F(s) = \prod_p \left\{ 1 + \sum_{\nu=2}^{\infty} a_\nu p^{-\nu s} \right\}.$$

Hence we obtain that

$$\sum_{n=1}^{\infty} \tau_3^2(n) n^{-s} = \{\zeta(s)\}^9 F(s),$$

where $F(s)$ is absolutely convergent for $\sigma > 1/2$. It follows at once, by Lemma 3.11, that

$$\sum_{n \le X} \tau_3^2(n) = \frac{a_3}{8!} X P_8(\log X) + O\left(X^{10/11}\right),$$

where

$$a_3 = F(1) = \prod_p \left(1 - 9p^{-2} + 16p^{-3} - 9p^{-4} + p^{-6}\right).$$

□

LEMMA 3.13. *We have*

$$\int_1^{N-1} \frac{t \log t}{N-t} dt = N\left(\log^2 N - \log N - \frac{\pi^2}{6} + 1\right) + O(\log N)$$

and

$$\int_1^{N-1} \frac{t \log^2 t}{N-t} dt = N\left(\log^3 N - 2\log^2 N - \left(\frac{\pi^2}{3} - 2\right)\log N + 2\zeta(3) - 2\right) + O(\log^2 N).$$

Proof. Expanding into a geometric series and integrate by parts, we have

$$\begin{aligned}
\int_1^{N-1} \frac{t \log t}{N-t} dt &= \sum_{m=1}^{\infty} \frac{1}{N^m} \int_1^{N-1} t^m \log t dt \\
&= N \log(N-1) \sum_{m=1}^{\infty} \frac{1}{m+1} \left(\frac{N-1}{N}\right)^{m+1} - N \sum_{m=1}^{\infty} \frac{1}{(m+1)^2} \left(\frac{N-1}{N}\right)^{m+1} + O(1) \\
&= N\left(\log^2 N - \log N - \frac{\pi^2}{6} + 1\right) + O(\log N).
\end{aligned}$$

This gives the first integral. The second integral is computed in a similar way. □

4. An analogue to a result of Lavrik

In this section we estimate the integral in (3.21) by trigonometric method of I.M. Vinogradov along the line of Lavrik, following Motohashi (section 3).

Let a/q be a term of the Farey series of order Ω, which is to be determined latter. Let

$$\frac{a'}{q'}, \frac{a}{q}, \frac{a''}{q''}$$

be consecutive terms of the Farey series and let $C(a/q)$ be the interval $\left[\frac{a'+a}{q'+q}, \frac{a+a''}{q+q''}\right]$. The interval $C(a/q)$ contains the fraction a/q with length bounded by

$$\left|C\left(\frac{a}{q}\right)\right| \leq \frac{2}{q\Omega}. \tag{4.1}$$

Let

$$U(N) = \int_0^1 \left||D(\alpha, N)|^2 - G_\Delta(\alpha, N)\right|^2 d\alpha$$

denote the integral in (3.21). We proceed to estimate $U(N)$. We have

$$\begin{aligned} U(N) &= \sum_{1\leq q\leq\Omega} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{C(a/q)} \left||D(\alpha, N)|^2 - G_\Delta(\alpha, N)\right|^2 d\alpha \\ &\leq 2 \sum_{1\leq q\leq\Omega} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{C(a/q)} \left||D(\alpha, N)|^2 - \left|F\left(\alpha, \frac{a}{q}, N\right)\right|^2\right|^2 d\alpha \\ &+ 2 \sum_{1\leq q\leq\Omega} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{C(a/q)} \left|G_\Delta(\alpha, N) - \left|F\left(\alpha, \frac{a}{q}, N\right)\right|^2\right|^2 d\alpha \\ &= 2U_1(N) + U_2(N), \end{aligned} \tag{4.2}$$

say. For $U_1(N)$, we have, from (3.16),

$$\left||D(\alpha, N)|^2 - \left|F\left(\alpha, \frac{a}{q}, N\right)\right|^2\right|^2 \ll (Nq+q^2)^{\frac{6}{5}+2\varepsilon}\left(1 + \left|\alpha - \frac{a}{q}\right|^2 N^2\right)\left(|D(\alpha, N)|^2 + \left|F\left(\alpha, \frac{a}{q}, N\right)\right|^2\right).$$

Thus, for $\alpha \in C(a/q)$, we have, by (4.1), that the above is bounded by

$$\left((N\Omega)^{\frac{6}{5}+2\varepsilon} + \Omega^{\frac{12}{5}+4\varepsilon} + \frac{N^{\frac{16}{5}+2\varepsilon}}{\Omega^2}\right)\left(|D(\alpha, N)|^2 + \left|F\left(\alpha, \frac{a}{q}, N\right)\right|^2\right),$$

and we get

$$\begin{aligned} U_1(N) &\ll \left((N\Omega)^{\frac{6}{5}+2\varepsilon} + \Omega^{\frac{12}{5}+4\varepsilon} + \frac{N^{\frac{16}{5}+2\varepsilon}}{\Omega^2}\right)\left\{\int_0^1 |D(\alpha, N)|^2 d\alpha + \sum_{1\leq q\leq\Omega} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_0^1 \left|F\left(\alpha, \frac{a}{q}, N\right)\right|^2 d\alpha\right\} \\ &\ll \left((N\Omega)^{\frac{6}{5}+2\varepsilon} + \Omega^{\frac{12}{5}+4\varepsilon} + \frac{N^{\frac{16}{5}+2\varepsilon}}{\Omega^2}\right) N \log^8 N. \end{aligned} \tag{4.3}$$

For $U_2(N)$, we have, by (3.14),

$$(4.4)\qquad U_2(N) \ll \sum_{1\le q\le \Omega} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{C(a/q)} \left| \sum_{1\le q'\le \Delta} \sum_{\substack{a'=1\\(a',q')=1\\a'q\ne aq'}}^{q'} \left| F\left(\alpha, \frac{a}{q}, N\right)\right|^2 \right|^2 d\alpha$$

$$+ \sum_{\Delta<q\le \Omega} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{C(a/q)} \left| F\left(\alpha, \frac{a}{q}, N\right)\right|^4 d\alpha$$

$$= U_3(N) + U_4(N),$$

say. By (3.13), we have

$$(4.5)\qquad U_4(N) \ll \frac{(N\log^2 N)^4}{\Omega} \sum_{\Delta<q\le\Omega} \frac{1}{q^4} \ll \frac{N^4\log^8 N}{\Omega\Delta^3}.$$

It remains to estimate $U_3(N)$. By partial summation, we can write $F(\alpha, a'/q', N)$ as

$$\frac{1}{q'}(A(a',q')\log^2 N + (B(a',q') - 2A(a',q'))\log N + 2A(a',q') - B(a',q')$$

$$+ C(a',q')) \sum_{1\le n\le N} e\left(\left(\alpha - \frac{a'}{q'}\right)n\right) - \frac{1}{q'}\int_1^N \left(\frac{2A\log\xi}{\xi} + \frac{B-2A}{\xi}\right) \sum_{1\le n\le \xi} e\left(\left(\alpha - \frac{a'}{q'}\right)n\right) d\xi.$$

Thus,

$$\left| F\left(\alpha, \frac{a'}{q'}, N\right)\right| \ll \frac{q'^{\varepsilon}\log^3 N}{q'\left|\sin\pi\left(\alpha - \frac{a'}{q'}\right)\right|}.$$

The function $F(\alpha, a'/q', N)$ has period 1 in α, and $|a/q - (a'/q' \pm 1)| \le 1/2$. Thus, $U_3(N)$ is at most

$$\ll \Delta^2 \log^{12} N \sum_{1\le q\le \Omega} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{C(a/q)} \sum_{1\le q'\le\Delta} \sum_{\substack{a'=-q'\\(a',q')=1\\0<\left|\frac{a'}{q'}-\frac{a}{q}\right|\le\frac12}}^{2q'} \frac{q'^{\varepsilon}}{q'^4\left|\sin\pi\left(\alpha - \frac{a'}{q'}\right)\right|^4} d\alpha.$$

By (3.15), we have, for $\alpha \in C(a/q)$,

$$\frac{1}{2}\left|\frac{a}{q} - \frac{a'}{q'}\right| \le \left|\alpha - \frac{a'}{q'}\right| \le \frac{3}{4}$$

for N sufficiently large. Hence,

$$U_3(N) \ll \Omega^2\Delta^2\log^{12} N \sum_{1\le q\le\Omega} \sum_{\substack{a=1\\(a,q)=1}}^{q} \sum_{1\le q'\le\Delta} q'^{\varepsilon} \sum_{\substack{a'=-q'\\(a',q')=1\\a'q\ne aq'}}^{2q'} \frac{1}{|aq' - qa'|^4} \ll \Omega^{2+\varepsilon}\Delta^2\log^{12} N \sum_{u=1}^{\infty} \frac{t(u)}{u^4},$$

where $t(u)$ is the number of integer solutions to $|aq' - qa'| = u$ in the range of summation. We have

$$t(u) \ll \Delta^2 \Omega$$

which yields

$$U_3(N) \ll \Omega^{3+\varepsilon} \Delta^4 \log^{12} N. \tag{4.6}$$

From (3.21), (4.2), (4.3), (4.4), (4.5) and (4.6), we get the inequality

$$\sum_{1 \le k \le N-1} (V(k,N) - S_\Delta(k,N))^2 \ll N^\varepsilon \left(N^{11/5} \Omega^{6/5} + \Omega^{12/5} N + \frac{N^{21/5}}{\Omega^2} + \Omega^3 \Delta^4 + \frac{N^4}{\Omega \Delta^3} \right).$$

We now take, for example,

$$\Omega = N^{25/38} \quad \text{and} \quad \Delta = N^{4/19}. \tag{4.7}$$

Then the requirement (3.15) is satisfied, and we have proved

LEMMA 4.1. *The inequality*

$$\sum_{1 \le k \le N-1} (V(k,N) - S_\Delta(k,N))^2 \ll N^{299/100}$$

holds for sufficiently large N.

5. Proof of the theorem

Let $Q(N)$ denote the sum on the left side of (2.1). We have

(5.1)

$$
\begin{aligned}
Q(N) = &\sum_{1\le \ell\le N} \sum_{\substack{1\le n_1,n_2\le N\\ n_1\equiv n_2 (\mathrm{mod}\ \ell)}} \tau_3(n_1)\tau_3(n_2)\\
&+ \frac{1}{4}N^2\log^4 N \sum_{1\le \ell\le N}\sum_{1\le b\le \ell} \tilde{A}(\ell,b)^2\\
&- N^2\log^3 N \sum_{1\le \ell\le N}\sum_{1\le b\le \ell} (\tilde{A}(\ell,b)^2 - \tilde{A}(\ell,b)\tilde{B}(\ell,b))\\
&+ N^2\log^2 N \sum_{1\le \ell\le N}\sum_{1\le b\le \ell} (\tilde{A}(\ell,b)^2 - 2\tilde{A}(\ell,b)\tilde{B} + \tilde{B}(\ell,b)^2)\\
&+ N^2\log^2 N \sum_{1\le \ell\le N}\sum_{1\le b\le \ell} (\tilde{A}(\ell,b)^2 - \tilde{A}(\ell,b)\tilde{B}(\ell,b) + \tilde{A}(\ell,b)\tilde{C}(\ell,b))\\
&+ 2N^2\log^2 N \sum_{1\le \ell\le N}\sum_{1\le b\le \ell} (\tilde{A}(\ell,b)^2 + \tilde{B}(\ell,b)^2 - 2\tilde{A}(\ell,b)\tilde{B}(\ell,b) - \tilde{B}(\ell,b)\tilde{C}(\ell,b) + \tilde{A}(\ell,b)\tilde{C}(\ell,b))\\
&+ N^2 \sum_{1\le \ell\le N}\sum_{1\le b\le \ell} (\tilde{A}(\ell,b)^2 + \tilde{B}(\ell,b)^2 + \tilde{C}(\ell,b)^2 - 2\tilde{A}(\ell,b)\tilde{B}(\ell,b) + 2\tilde{A}(\ell,b)\tilde{C}(\ell,b) - 2\tilde{B}(\ell,b)\tilde{C}(\ell,b))\\
&- N\log^2 N \sum_{1\le \ell\le N}\sum_{1\le b\le \ell} \tilde{A}(\ell,b) \sum_{\substack{1\le n\le N\\ n\equiv b(\ell)}} \tau_3(n)\\
&+ 2N\log N \sum_{1\le \ell\le N}\sum_{1\le b\le \ell} (\tilde{A}(\ell,b) - \tilde{B}(\ell,b)) \sum_{\substack{1\le n\le N\\ n\equiv b(\ell)}} \tau_3(n)\\
&+ 2N \sum_{1\le \ell\le N}\sum_{1\le b\le \ell} (\tilde{A}(\ell,b) - \tilde{B}(\ell,b) + \tilde{C}(\ell,b)) \sum_{\substack{1\le n\le N\\ n\equiv b(\ell)}} \tau_3(n)\\
&= Q_1(N) + \cdots + Q_{10}(N),
\end{aligned}
$$

say. We start with evaluating $Q_1(N)$, which is the longest of the ten. We have

$$Q_1(N) = N\sum_{1\le n\le N}\tau_3^2(n) + 2\sum_{1\le \ell\le N-1}\ \sum_{1\le u\le (N-1)/\ell}\ \sum_{1\le n\le N-u\ell}\tau_3(n)\tau_3(n+u\ell) \tag{5.2}$$
$$= N\sum_{1\le n\le N}\tau_3^2(n) + 2\sum_{1\le k\le N-1}V(k,N)\tau(k),$$

where $V(k,N)$ is given by (3.20). Here we have, by Lemma 3.12,

$$\sum_{n\le N}\tau_3^2(n) = \frac{a_3}{8!}NP_8(\log N) + O\left(N^{10/11}\right) \tag{5.3}$$

with a_3 and $P_8(\log N)$ given in that lemma. Now, by Lemma 4.1,

$$\sum_{1\le k\le N-1}V(k,N)\tau(k) = \sum_{1\le k\le N-1}S_\Delta(k,N)\tau(k) \tag{5.4}$$
$$+O\left\{\left(\sum_{1\le k\le N-1}\tau^2(k)\right)^{1/2}\left(\sum_{1\le k\le N-1}(V(k,N)-S_\Delta(k,N))^2\right)^{1/2}\right\}$$
$$= \sum_{1\le k\le N-1}S_\Delta(k,N)\tau(k) + O\left(N^{599/300}\right) = Q_{11}(N) + O\left(N^{599/300}\right),$$

say. We now calculate $Q_{11}(N)$. By (3.19), (3.18), and (3.22), we have

$$Q_{11}(N) = \sum_{j=1}^{6}\sum_{1\le q\le\Delta}q^{-2}w_j(q)\sum_{1\le k\le N-1}\tau(k)c_q(k)T_j(k,N).$$

If $q=1$, then

$$c_1(k)=1,\ A(1)=1,\ B(1)=3\gamma,\ C(1)=3\gamma^2,$$

and, hence,

$$\begin{aligned} w_1(1) &= \frac{1}{4},\\ w_2(1) &= \frac{1}{2}(3\gamma-1),\\ w_3(1) &= (1-3\gamma)^2,\\ w_4(1) &= -\frac{1}{2}(1-3\gamma+3\gamma^2),\\ w_5(1) &= (3\gamma-1)(1-3\gamma-3\gamma^2),\\ w_6(1) &= (1-3\gamma+3\gamma^2)^2. \end{aligned} \tag{5.5}$$

Thus,

$$Q_{11}(N) = \sum_{j=1}^{6}w_j(1)\sum_{1\le k\le N-1}\tau(k)T_j(k,N) \tag{5.6}$$
$$+\sum_{j=1}^{6}\sum_{1<q\le\Delta}q^{-2}w_j(q)\sum_{1\le k\le N-1}\tau(k)c_q(k)T_j(k,N).$$

To calculate the k-summations, we need to compute the following sums.

$$
\begin{aligned}
H_1(N) &= \sum_{1\le k\le N-1} \tau(k)\log(N-k),\\
H_2(N) &= \sum_{1\le k\le N-1} \tau(k)\log^2(N-k),\\
H_3(X) &= \sum_{1\le k\le X} \tau(k)c_q(k),\\
H_4(N) &= \sum_{1\le k\le N-1} \tau(k)c_q(k)\log(N-k),\\
H_5(N) &= \sum_{1\le k\le N-1} \tau(k)c_q(k)\log^2(N-k),\\
H_6(N) &= \sum_{1\le k\le N-1} k\tau(k)\log(N-k),\\
H_7(N) &= \sum_{1\le k\le N-1} k\tau(k)\log^2(N-k),\\
H_8(X) &= \sum_{1\le k\le X} k\tau(k)c_q(k),\\
H_9(N) &= \sum_{1\le k\le N-1} k\tau(k)c_q(k)\log(N-k),\\
H_{10}(N) &= \sum_{1\le k\le N-1} k\tau(k)c_q(k)\log^2(N-k).
\end{aligned}
\tag{5.7}
$$

Assume $q>1$. We now compute the first sum in (5.7). By partial summation, we have

$$H_1(N) = \int_1^{N-1} \frac{t}{N-t}\log t dt + (2\gamma-1)\int_1^{N-1}\frac{t}{N-t}dt + O(N^{1/2}\log N).$$

By the first part of Lemma 3.13, this is equal to

$$N\left(\log^2 N - \log N - \frac{\pi^2}{6}+1\right) + (2\gamma-1)(N\log N - N) + O(N^{1/2}\log N).$$

Thus,

$$H_1(N) = N\log^2 N + (2\gamma-2)N\log N + \left(\frac{\pi^2}{6}-2\gamma\right)N + O(N^{1/2}\log N).$$

Similar, by both parts of Lemma 3.13, we get

$$
\begin{aligned}
H_2(N) &= \int_1^{N-1}\frac{1}{N-t}\left(t\log^2 t + (2\gamma-2)t\log t + \left(\frac{\pi^2}{6}-2\gamma\right)t + O(t^{1/2}\log t)\right)dt\\
&= N\log^3 N + (2\gamma-4)N\log^2 N + \left(4-4\gamma-\frac{\pi^2}{6}\right)N\log N\\
&+ \left(2\zeta(3) - 2 - (2\gamma-2)\left(\frac{\pi^2}{6}-1\right) - \frac{\pi^2}{6} + 2\gamma\right)N + O\left(N^{1/2}\log^2 N\right).
\end{aligned}
$$

We now estimate $H_3(X)$. We have

$$H_3(X) = \sum_{1\le k\le X}\sum_{d|k} c_q(k) = \sum_{1\le d\le X}\sum_{1\le m\le X/d} c_q(dm).$$

By Lemma 3.10, the inner sum is $\ll qd^{-1}X^{1/2}\log^2 X$. Thus,

$$H_3(X) \ll qX^{1/2}\log^2 X \sum_{1\le d\le X} d^{-1} \ll qX^{1/2}\log^3 X.$$

Using this we get, by partial summation,

$$H_4(N) = O(qN^{1/2}\log^4 N)$$

and

$$H_5(N) = O(qN^{1/2}\log^5 N).$$

By partial summation we can easily obtain

LEMMA 5.1.

$$H_6(N) = \frac{1}{2}(N-1)^2\log^2(N-1) + \lambda_1(N-1)^2\log(N-1) + \lambda_2(N-1)^2 + O(N^{3/2}\log N),$$

$$H_7(N) = \frac{1}{2}(N-1)^2\log^3(N-1) + \lambda_3(N-1)^2\log^2(N-1) + \lambda_4(N-1)^2\log(N-1) + \lambda_5(N-1)^2 + O(N^{3/2}\log^3 N),$$

$$H_8(N) = O(N^{3/2}\log^3 N),$$

$$H_9(N) = O(N^{3/2}\log^4 N),$$

$$H_{10}(N) = O(N^{3/2}\log^4 N),$$

with numerical constants λ_j's.

Here we have

$$\lambda_1 = \gamma - 1/2,$$
$$\lambda_2 = \frac{\pi^2}{12} - \frac{1}{2}\gamma - \frac{3}{4},$$
$$\lambda_3 = \gamma - 5/4,$$

etc.

Collecting the $w_j(1)$'s from (5.5) and the H_j's above, we deduce the following

LEMMA 5.2. *There is an explicit polynomial $P_5(\log N)$ of degree 5 in $\log N$ such that the $q = 1$ contribution in $Q_{11}(N)$ from (5.6) is given by*

$$\sum_{j=1}^{6} w_j(1) \sum_{1\le k\le N-1} \tau(k)T_j(k,N) = N^2P_5(\log X) + O(N^{3/2}\log^3 N).$$

Moreover, the $q > 1$ contributions in $Q_{11}(N)$ from (5.6) is at most $O(N^{3/2})$, and, consequently, from (5.2), (5.3), (5.4), and (5.6), we obtain that

$$Q_1(N) = N^2P_8(\log N) + O(N^{599/300}).$$

With more effort, though tedious in details, one can calculate similar asymptotic expansions for $Q_2(N)$ to $Q_{10}(N)$ in (5.1). However, for our purpose, it suffices to bound the sums Q_2-Q_{10} and show that they are smaller than the leading term $N^2\log^8 N$. Indeed, by (3.5) and orthogonality of the Ramanujan sum $c_q(b)$, we have that

$$Q_2(N), \cdots, Q_{10}(N) \ll N^2\log^6 N. \tag{5.8}$$

We demonstrate one such bound for $Q_2(N)$–the other bounds can be obtained similarly. Suppose first that $q = 1$. We have, in this case, $\tilde{A}(\ell, b) = \ell^{-1}$ for any b, and hence

$$\sum_{1\le\ell\le N}\sum_{1\le b\le\ell} \tilde{A}(\ell, b)^2 = \sum_{1\le\ell\le N}\sum_{1\le b\le\ell} \ell^{-2} \ll \log N. \tag{5.9}$$

Assume next $q_1, q_2 > 1$. Suppose $(q_1, q_2) = 1$. Then

$$\sum_{1\le b\le\ell} c_{q_1}(b)c_{q_2}(b) = \sum_{1\le b\le\ell} c_{q_1q_2}(b) \ll q_1q_2.$$

From this and (3.5), we get

$$\begin{aligned}\sum_{1\le b\le\ell} \tilde{A}^2(\ell, b) &= \ell^{-2}\sum_{q_1|\ell}\sum_{q_2|\ell} q_1^{-1}q_2^{-1}\sum_{1\le b\le\ell} c_{q_1}(b)c_{q_2}(b)\tau(q_1)\log^2 q_1\tau(q_2)\log^2 q_2\\ &= \ell^{-2}\sum_{q_1|\ell}\tau(q_1)\log^2 q_1\sum_{q_2|\ell}\tau(q_2)\log^2 q_2 \ll \ell^{-2}\tau^4(\ell)\log^4\ell \ll \ell^{-1}\end{aligned}$$

and, hence,

$$\sum_{1\le\ell\le N}\sum_{1\le b\le\ell} \tilde{A}(\ell, b)^2 \ll \sum_{1\le\ell\le N} \ell^{-1} \ll \log N. \tag{5.10}$$

It remains to consider the case where $(q_1, q_2) > 1$. Let $q_0 = [q_1, q_2]$. By orthogonality of $c_q(b)$ we have that

$$\sum_{1\le b\le q_0} c_{q_1}(b)c_{q_2}(b) = \begin{cases} q_0\varphi(q_0), & \text{if } q_1 = q_2,\\ 0, & \text{otherwise.}\end{cases}$$

Thus,

$$\sum_{1\le b\le\ell} c_{q_1}(b)c_{q_2}(b) \ll \begin{cases} \ell\varphi(q_0), & \text{if } q_1 = q_2,\\ q_1q_2, & \text{otherwise,}\end{cases}$$

which gives

$$\sum_{1\le\ell\le N}\sum_{1\le b\le\ell} \tilde{A}^2(\ell, b) \ll \begin{cases} \sum_{1\le\ell\le N} \ell^{-1}\tau(\ell) \ll \log^2 N, & \text{if } q_1 = q_2,\\ \sum_{1\le\ell\le N} \ell^{-1} \ll \log N, & \text{if } q_1 \ne q_2.\end{cases}$$

This, together with (5.9) and (5.10), give that $Q_2(N)$ is at most $O(N^2\log^6 N)$, verifying (5.8) for $Q_2(N)$.

As mentioned before, the estimates in (5.8) are crude simply for the purpose of showing they do not contribute to the leading term. It is possible, by procedures analogous to the computations for $Q_1(N)$ and $\sum_k W_q(k, N)$ demonstrated in the proof, to compute explicitly a polynomial $P_6(\log N)$ of degree 6 in $\log N$ such that

$$Q_2(N) + \cdots + Q_{10}(N) = N^2P_6(\log N) + O(N^{599/300}).$$

We conclude, therefore, that $Q(N)$, which is the left-hand side of (2.1), is given by

$$N^2 P_8(\log N) + O(N^{2-1/300}),$$

which gives the right-hand side of (2.1). This completes the proof of the theorem.

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