A note on the zeros of the derivative of the Riemann zeta function near the critical line

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The purpose of this note is to give an explicit numerical lower bound for the positive proportion of zeros of the first derivative of the Riemann zeta function near the critical line, refining the main result of Zhang [6].

Let $\rho' = \beta' + i\gamma'$ denote the zeros of $\zeta'(s)$, $s = \sigma + it$. The distribution of zeros of $\zeta'(s)$ is intimately related to those of $\zeta(s)$. For $\nu \in \mathbb{R}$, define

$$m^-(\nu) = \liminf_{T\to\infty} \frac{1}{N_1(T)} \sum_{\substack{\beta' \le 1/2+\nu/\log T \\ 0<\gamma'\le T}} 1, \tag{1.2}$$

where $N_1(T)$ is the number of zeros of zeta prime with $0 < \gamma' \le T$. The behavior of the function $m^-(\nu)$ determines the horizontal distribution of the zeros of zeta prime near the critical line. In 2001 Zhang [6] showed, unconditionally, that $m^-(\nu) > 0$ for ν sufficiently large. We give a quantitative lower bound, complementing the result in [6, Theorem 1].

THEOREM 0.1. *For $\nu > 10^{22}$, we have*

$$m^-(\nu) > 10^{-83}. \tag{1}$$

We give a sketch of proof. We follow Heath-Brown's note 9.26 in Titchmarsh [4]. Let

$$Q(t) = S(t) - \frac{1}{\pi} \sum_{p < T^{\frac{1}{20k}\left(a-\frac{1}{2}\right)}} \frac{\sin(t\log p)}{\sqrt{p}}, \tag{2}$$

$$P(t) = \frac{1}{\pi}\Im \sum_{p < T^{\frac{1}{20k}\left(a-\frac{1}{2}\right)}} p^{-\frac{1}{2}-it}(p^{-ih}-1), \tag{3}$$

and

$$U(t) = Q(t+h) - Q(t), \tag{4}$$

so that $S(t+h) - S(t) = U(t) + P(t)$.

By Montgomery, Vaughan, and Vaaler's generalization [3, Corollary, page 36] of Hilbert's inequality, we deduce that

$$\int_T^{T+H} \left|\sum_r a_r r^{-it}\right|^2 dt \le H\sum_r |a_r|^2 + 336\theta \sum_r r|a_r|^2, \tag{5}$$

for some θ, $-1 \le \theta \le 1$. It follows from this that

$$\left| \int_T^{T+H} \left(\sum_n a_n n^{-it} \right) \overline{\left(\sum_n b_n n^{-it} \right)} - H \sum_n a_n \overline{b_n} \right| \le 336 \left(\sum_n n|a_n|^2 \right)^{1/2} \left(\sum_n n|b_n|^2 \right)^{1/2} . \quad (6)$$

From the above inequality, it can be shown that

$$\left| \int_T^{T+H} \{f(t)\}^2 dt - \frac{1}{2} H \sum_p |a_p|^2 \right| \le 672 \sum_p p|a_p|^2 \quad (7)$$

and

$$\left| \int_T^{T+H} \{f(t)\}^4 dt - \frac{3}{2} H \sum_{p,q} |a_{pq}|^2 \right| \le 4032 \left(\sum_p p|a_p|^2 \right)^2 , \quad (8)$$

where $f(t)$ denotes either the real or imaginary part of $\sum_p a_p p^{-it}$. For the derivation, see the proofs of Lemmas 1 and 2 of Tsang [5].

In a way analogous to the proof of Theorem 4 of Tsang [5], we get by applying (7) and (8) to $f(t) = P(t)$ that

$$\left| \int_T^{T+H} \{P(t)\}^2 dt \right| \le \pi^{-2} H \log(2 + h \log T) \quad (9)$$

and

$$\left| \int_t^{T+H} \{P(t)\}^4 dt - \frac{3}{\pi^4} H \log^2(2 + h \log T) \right| \le 672 H \log(2 + h \log T). \quad (10)$$

From Lemma 7 on page 440 of Karatsuba and Korolëv [2], we deduce that

$$\int_T^{T+H} |Q(t)|^2 dt \le \left(\frac{e^{37}}{10^{-12}\pi^2} \right) H < 10^{29} H \quad (11)$$

and

$$\int_T^{T+H} |Q(t)|^4 dt \le \left(\frac{e^{37} 4}{10^{-12}\pi^2} \right)^2 H < 10^{58} H. \quad (12)$$

Hence, by (9) and (11),

$$\int_T^{T+H} \{S(t+h) - S(t)\}^2 dt = \int_T^{T+H} \{U(t) + P(t)\}^2 dt \quad (13)$$

$$\le \int_T^{T+H} \{U(t)\}^2 dt + \int_T^{T+H} \{P(t)\}^2 dt + 2 \left(\int_T^{T+H} \{P(t)\}^2 dt \right)^{1/2} \left(\int_T^{T+H} \{U(t)\}^2 dt \right)^{1/2} \quad (14)$$

$$< 10^{30} H + \pi^{-2} H \log(2 + h \log T) + 2 \left(\pi^{-2} H \log(2 + h \log T) \right)^{1/2} \left(10^{30} H \right)^{1/2} \quad (15)$$

$$< H(10^{30} + \pi^{-2} \log(2 + h \log T)) + 10^{15} H \log^{1/2}(2 + h \log T). \quad (16)$$

Similarly, by (10) and (12) and Hölder inequality, we get

$$\int_T^{T+H} \{S(t+h)-S(t)\}^4 dt = \int_T^{T+H} \{U(t)+P(t)\}^4 dt \tag{17}$$

$$\leq \int_T^{T+H} \{U(t)\}^4 dt + \int_T^{T+H} \{P(t)\}^4 dt + 4\left(\int_T^{T+H} \{U(t)\}^4\right)^{1/4} \left(\int_T^{T+H} \{P(t)\}^4\right)^{3/4} \tag{18}$$

$$+ 4\left(\int_T^{T+H} \{U(t)\}^4\right)^{3/4} \left(\int_T^{T+H} \{P(t)\}^4\right)^{1/4} + 6\left(\int_T^{T+H} \{U(t)\}^4\right)^{1/2} \left(\int_T^{T+H} \{P(t)\}^4\right)^{1/2} \tag{19}$$

$$< H\left(10^{60} + \frac{3}{\pi^4}\log^2(2+h\log T)\right) + 10^{16} H \log^{3/2}(2+h\log T). \tag{20}$$

For $2\pi M \leq h \log T \leq 4\pi M$, the inequalities

$$\left|\int_T^{T+H} \{S(t+h)-S(t)\}^2 dt - H(10^{30} + \pi^{-2}\log(2+h\log T))\right| \leq 10^{15} H \log^{1/2}(2+h\log T) \tag{21}$$

and

$$\left|\int_T^{T+H} \{S(t+h)-S(t)\}^4 dt - H\left(10^{60} + \frac{3}{\pi^4}\log^2(2+h\log T)\right)\right| \leq 10^{16} H \log^{3/2}(2+h\log T) \tag{22}$$

hold if

$$M > e^{10^{36}}. \tag{23}$$

which we henceforth assume. With this value of M and taking $H = T$, (21) and (22) yield

$$\int_T^{2T} |S(t+h)-S(t)|^2 dt \geq T \tag{24}$$

and

$$\int_T^{2T} |S(t+h)-S(t)|^4 dt \leq 10^{36} T \tag{25}$$

for T large. By Hölder inequality we have

$$\int_T^{2T} |S(t+h)-S(t)|^2 dt \leq \left(\int_T^{2T} |S(t+h)-S(t)| dt\right)^{\frac{2}{3}} \left(\int_T^{2T} |S(t+h)-S(t)|^4 dt\right)^{\frac{1}{3}}, \tag{26}$$

so that

$$\int_T^{2T} |S(t+h)-S(t)| dt \geq \frac{\left(\int_T^{2T} |S(t+h)-S(t)|^2 dt\right)^{3/2}}{\left(\int_T^{2T} |S(t+h)-S(t)|^4 dt\right)^{1/2}} \geq \frac{T^{3/2}}{(10^{36})^{1/2} T^{1/2}} = 10^{-18} T. \tag{27}$$

We have that

$$N(t) = L(t) + S(t) \quad (t > t_0) \tag{28}$$

with

$$L(t) = \frac{t}{2\pi}\log t - \frac{1+\log 2\pi}{2\pi} t + \frac{7}{8} + O\left(\frac{1}{t}\right), \tag{29}$$

we obtain

$$L(t+h)-L(t)=\frac{1}{2\pi}\left(t\log\frac{t+h}{t}+h\log(t+h)\right)-\frac{1+\log 2\pi}{2\pi}h+O(1/t)=\frac{h\log T}{2\pi}+O\left(\frac{1}{\log T}\right), \tag{30}$$

and hence

$$S(t+h)-S(t)=N(t+h)-N(t)-\frac{h\log T}{2\pi}+O\left(\frac{1}{\log T}\right), \tag{31}$$

for $T\le t\le 2T$, whence

$$\int_T^{2T}\left|N(t+h)-N(t)-\frac{h\log T}{2\pi}\right|dt=\int_T^{2T}|S(t+h)-S(t)|dt+O\left(\frac{T}{\log T}\right)\ge 10^{-19}T. \tag{32}$$

We proceed to write $h=2\pi\lambda/\log T$, $1\le\lambda\le 2$, and

$$\delta(t,\lambda)=N\left(t+\frac{2\pi\lambda}{\log T}\right)-N(t)-\lambda, \tag{33}$$

so that

$$N(t+h)-N(t)-\frac{h\log T}{2\pi}=\sum_{m=0}^{M-1}\delta\left(t+\frac{2\pi m\lambda}{\log T},\lambda\right). \tag{34}$$

Thus,

$$10^{-19}T\le\int_T^{2T}\left|N(t+h)-N(t)-\frac{h\log T}{2\pi}\right|dt \tag{35}$$

$$=\sum_{m=0}^{M-1}\int_T^{2T}\left|\delta\left(t+\frac{2\pi m\lambda}{\log T},\lambda\right)\right|=\sum_{m=0}^{M-1}\int_{T+\frac{2\pi m\lambda}{\log T}}^{2T+\frac{2\pi m\lambda}{\log T}}|\delta(t,\lambda)|dt \tag{36}$$

$$=\int_T^{2T}|\delta(t,\lambda)|dt, \tag{37}$$

and hence

$$\int_T^{2T}|\delta(t,\lambda)|dt\ge 10^{-19}T \tag{38}$$

uniformly for $1\le\lambda\le 2$. Let I denote the subset of $[T,2T]$ on which $N\left(t+\frac{2\pi\lambda}{\log T}\right)=N(t)$, then

$$|\delta(t,\lambda)|\le\begin{cases}\delta(t,\lambda)+2\lambda, & (t\in I),\\ \delta(t,\lambda)+2\lambda-2, & (t\in[T,2T]-I),\end{cases} \tag{39}$$

so that (38) yields

$$10^{-19}T\le\int_T^{2T}|\delta(t,\lambda)|dt=\int_T^{2T}\delta(t,\lambda)dt+(2\lambda-2)T+2m(I), \tag{40}$$

where $m(I)$ is the measure of I. However,

$$\delta(t,\lambda)=N\left(t+\frac{2\pi\lambda}{\log T}\right)-N(t)-\lambda=S\left(t+\frac{2\pi\lambda}{\log T}\right)-S(t)+O\left(\frac{1}{\log T}\right) \tag{41}$$

and hence

$$\int_T^{2T} \delta(t,\lambda)dt = \int_T^{2T} \left(S\left(t + \frac{2\pi\lambda}{\log T}\right) - S(t) \right) dt + O\left(\frac{T}{\log T}\right) = O(\log T), \tag{42}$$

whence

$$10^{-19}T \le (2\lambda - 2)T + 2m(I) \tag{43}$$

and

$$m(I) \ge \frac{1}{2}\left(10^{-19} - 2\lambda + 2\right) T > 10^{-20}T, \tag{44}$$

if we choose $\lambda = 1 + 10^{-20} > 1$. Thus, if

$$S = S_1 = \left\{ n : T \le \gamma_n \le 2T, \gamma_{n+1} - \gamma_n \ge \frac{2\pi\lambda}{\log T} \right\}, \tag{45}$$

then

$$10^{-20}T \le m(I) \le \sum_t \frac{2\pi\lambda}{\log T} \le \sum_{n\in S} (\gamma_{n+1} - \gamma_n) + O(1), \tag{46}$$

so that

$$\left(10^{-20}\right)^2 T^2 \le \left\{ \sum_{n\in S} (\gamma_{n+1} - \gamma_n) \right\}^2 \le (\#S) \left(\sum_{n\in S} (\gamma_{n+1} - \gamma_n)^2 \right). \tag{47}$$

By Theorem 1 of Fujii [1], we have

$$\sum_{\gamma_n \le T} (\gamma_{n+1} - \gamma_n)^2 \le 9 \frac{2\pi T}{\log T/2\pi}. \tag{48}$$

Inserting this into (47), we get

$$\#S > 10^{-39} N(T) \tag{49}$$

showing that

$$\frac{\gamma_{n+1} - \gamma_n}{2\pi/\log \gamma_n} \ge 1 + 10^{-20} \tag{50}$$

holds for a positive proportion of n.

Let $c_1 = 10^{-39}$, $c_2 = 10^{-21}$ and $\nu = 10^{22}$. Then $c_1 > 0$ and c_2 and ν satisfy

$$0 < c_2 < \lambda - 1 \tag{3.2}$$

and

$$\left(\frac{\nu}{\nu + 2\pi\lambda} \right)^2 > \frac{\lambda + 1}{2\lambda}. \tag{3.3}$$

We choose

$$c = 5 \cdot 10^{-82} \tag{51}$$

so that

$$0 < c < \left(1 + \frac{\nu}{\pi\lambda}\right)^{-1} \left(c_1 - 4c - \frac{4c}{c_2} \right) \tag{52}$$

in [6, equation (3.15)] is satisfied. By the proof of Theorem 1 in [6], we conclude that

$$\sum_{\substack{\beta'<1/2+\nu/\log T\\ 0<\gamma'<T}} 1 > cT\log T \tag{53}$$

holds for sufficiently large T, and, consequently,

$$m^{-}(\nu) > 10^{-83} \tag{54}$$

for all $\nu > 10^{22}$.

References

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