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Ricci Flow, 3-manifolds & physics

-> Topological classification of compact 3-mflds (Thurston)
-> Ricci flow
paths in space of metrics on an n-mfld (Hamilton)

$$\frac{\partial}{\partial t} g_{ij}(t) = -2\,\mathrm{Ric}(g_{ij}(t))$$

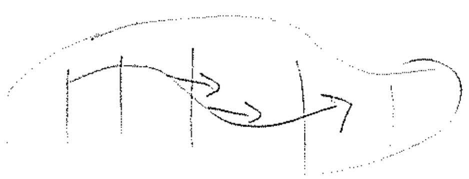
Non-linear heat equation
Dispersive solution

high curvature regions

Reformulation of Ricci flow as a gradient flow (Perelman)

$$\mathcal{F} = \int_M (R + |\nabla f|^2)\, e^{-f}\, dV$$

f – smooth function on M
R – scalar curvature



math.DG/0211159

$\delta \mathcal{F}(v_{ij}, h)$

$$v_{ij} = \delta g_{ij}$$
$$h = \delta f$$
$$v = g^{ij} v_{ij}$$

$$\int_M e^{-f} \left(-v_{ij} (R_{ij} + \nabla_i \nabla_j f) + \left(\frac{v}{2} - h\right)\left(2\Delta f - |\nabla f|^2 + R\right)\right)$$

Note: if $dm = e^{-f} dV$ held fixed, then $\frac{v}{2} = h$

Fix dm set $f = \log\left(\frac{dV}{dm}\right)$ D. Sullivan SUNY Lecture

$$\mathcal{F}^m = \int_M (R + |\nabla f|^2)\, dm$$

gradient flow: $\frac{\partial}{\partial t} g_{ij} = -2(R_{ij} + \nabla_i \nabla_j f)$

turns out: "mod" diffeomorphism, this is exactly Ricci flow

Existence of solutions is not always guaranteed (depends on dm), but if they exist, soln is indep of dm, up to diffeo

→ Physics

2D QFT w/ based on arbitrary Riemannian mflds

Given M, g_{ij}

$\mathrm{Maps}(\Sigma, M)$, $S(\varphi) = \int_\Sigma \varphi^*(g_{ij})$

$$\int_{Maps(\Sigma, M)} e^{-S(\varphi)} \boxed{\ldots}\, d\varphi$$

measure

(D. Friedan)

Renormalization (classically) governed by Ricci flow

n.b. RG actually a semigroup

Q Ricci flow on sigma or M?

A Change metric on M, rescale on Σ

Dim M=2 (Hamilton, mid 1980's)

Curvature

$$TM \otimes TM \longrightarrow TM$$

$$\nabla_Y X = \nabla_X Y = Z$$

$$R(X,Y) = \nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - \nabla_{[X,Y]}$$

$$Rm(X,Y,Z,W) = \langle R(X,Y)Z, W \rangle$$

skew skew

sym

Ric = contract the middle variables

$$Ric(e_i, e_j) = \sum_{k=1}^{\dim} Rm(e_i, e_k, e_k, e_j)$$

Scalar Curvature

$$R = \sum g^{ij} R_{ij}$$

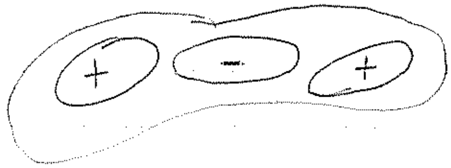
If $g_{ij}(t)$ is evolving by Ricci flow, then $R =$ scalar curvature satisfies

$$\frac{\partial}{\partial t} R = \Delta R + 2R^2$$

Non-linear heat equation

$\underline{\dim M = 2}$

$R_{ij} = R g_{ij}$ (Hamilton, mid 1980's B. Chow, ..)



$\underline{Th^m}$ Given a Riemannian metric on a compact surface consider rescaled Ricci flow

$$g_{ij}(t)' = -2R_{ij} + \bar{R} g_{ij}$$

where $\bar{R}$ = average scalar curvature

It exists for all time, and the $t \to \infty$ limit is a metric of constant curvature.

Non-rescaled: $g = 0$, vol $\to 0$ in finite time
$g = 1$, vol $\to$ finite
$g > 1$, vol $\to \infty$

(entire Ricci flow
$$g_{ij}(t) = e^{f(t)} g_{ij}(0)$$)

$\underline{\text{Classical } Th^m}$ The conformal class of any metric on a compact surface has a unique constant curvature representation.

$M = X / \Gamma$

$X =$ simply connected surface with a constant curvature metric: i.e. $X = S^2, \mathbb{R}^2, \mathbb{H}^2$

<u>geometric</u>: $X = H \backslash G$
$G = \mathrm{Isom}(X)$

<u>Three-manifolds</u> (Thurston)
$X = H \backslash G$ $\qquad G = \mathrm{Isom}(X)$, acts transitively

More generally, we can look at X/Γ s.t. $\mathrm{vol}(X/\Gamma) < \infty$

<u>Eight Geometries</u> Six of these geometries have circle ~~fibrations~~ Seifert fibrations

$$\begin{pmatrix} z \mapsto e^{2\pi i k/n} z \\ \theta \mapsto \theta + 2\pi/n \end{pmatrix}$$

$$\begin{tikzcd} S^1 \arrow[r] & M \arrow[d] \\ & \Sigma \end{tikzcd}$$

$e =$ top. class of S^1-bundle
$\chi =$ euler number of base

	$\chi < 0$	$\chi = 0$	$\chi > 0$
$e = 0$	$S^2 \times \mathbb{R}$	$\mathbb{R}^3$	$\mathbb{H}^2 \times \mathbb{R}$
$e \neq 0$	S^3	Nil	$\widetilde{Sl_2(\mathbb{R})}$

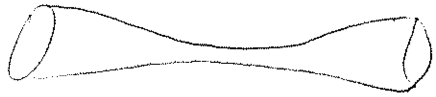
Hopf fibration

2-geometries have T^2 "fibrations" over 1-dim
Nil, Sol
3-geometries have constant curvature
S^3, $\mathbb{R}^3$, $\mathbb{H}^3$

$$Nil \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \qquad Sol \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$$

(M, g_{ij}) M compact $\dim M = 3$

Let gradient flow commence.
$S^3 \to$ pt in finite time



neck pinches off in finite time

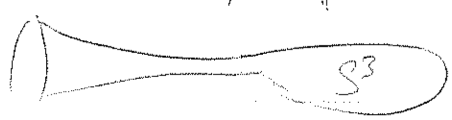
$$S^2 \times I = S^2 \times D^1$$
$$\partial(S^2 \times D^1) = S^2 \times S^0 = \partial(D^3 \times S^0)$$

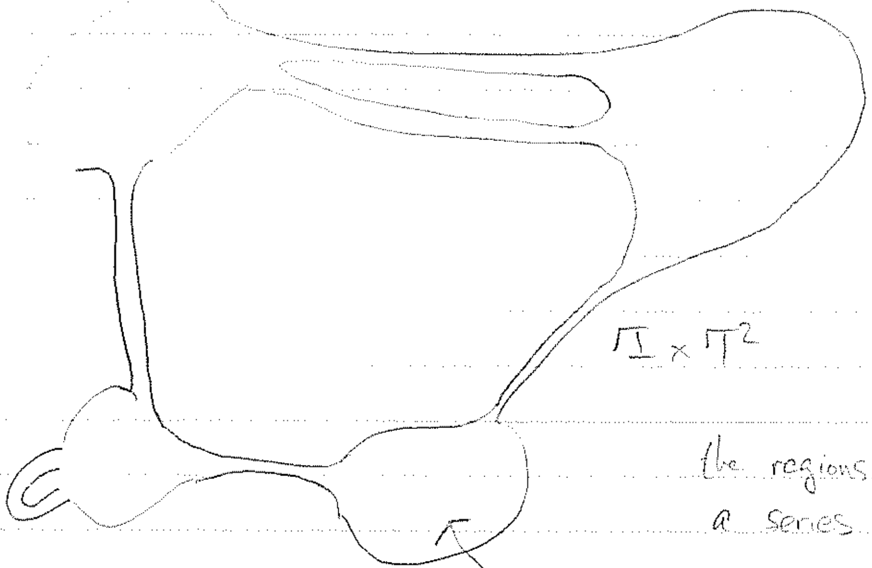
Perelman
- There are the only two types of singularities in finite time
- No accumulation points of singularity times

Hypothesis: No embedded $\mathbb{RP}^2$'s

Comment: embedded $\mathbb{RP}^2$ has twisted normal bundle
embedded $\mathbb{RP}^2 \Rightarrow$ connected sum

Surgeries / extinctions happen at a discrete set of evolution times
$t_1, t_2, t_3, \ldots$
unknown if the number is finite
for $t \gg 0$, only thing that happens

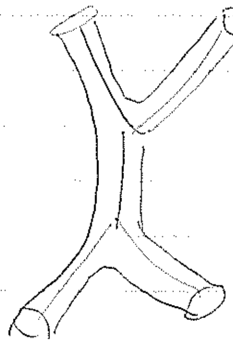




the regions are connected by a series of tubes

"geometric" as $t \to \infty$

if geometric piece is <u>not</u> $\mathbb{H}^3/\Gamma$
then it collapses along S^1, T^2 or T^3/Γ "fibration"



String Theory

$$2D \text{ QFT} \xrightarrow{S^1 \text{ bundle}} \text{String Theory}$$