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taken by M. Gary

Ricci Flow, 3-manifolds, & Physics

→ Topological classification of compact 3-manifolds (Thurston)

→ Ricci flow: paths in space of metrics on n-manifolds (Hamilton)

$$\frac{\partial}{\partial t} g_{ij}(t) = -2 R_{ij}(g_{ij}(t))$$

non-linear heat equation, dispersive solution

→ Reformulation of Ricci flow as a gradient flow (Perelman)

$$\mathcal{F} = \int_M (R + |\nabla f|^2) e^{-f} dV \qquad f = \text{smooth function on } M$$

$$V_{ij} = \delta g_{ij} \qquad h = \delta f \qquad V = g^{ij} V_{ij}$$

$$\delta \mathcal{F}(V_{ij}, h) = \int_M e^{-f}\left(-V_{ij}(R_{ij} + \nabla_i \nabla_j f) + \left(\frac{V}{2} - h\right)(2\Delta f - |\nabla f|^2 + R)\right)$$

<u>Note)</u> if $dm = e^{-f} dV$ held fixed, then $\frac{V}{2} = h$

<u>Fix dm:</u> set $f = \log\left(\frac{dV}{dm}\right)$

$$\mathcal{F}^m = \int_M (R + |\nabla f|^2) dm$$

gradient flow: $\frac{\partial}{\partial t} g_{ij} = -2(R_{ij} + \nabla_i \nabla_j f)$

turns out: mod diffeomorphism, this is exactly Ricci flow.

Existence of solutions is not always guaranteed (depends on dm), but if they exist, solution is indep. of dm, up to diffeomorphism.

→ **Physics**: 2-dim Quantum field theories based on arbitrary Riemannian manifolds (D. Friedan)

$$M, g_{ij}$$

$$\mathrm{Maps}(\Sigma, M) \qquad S(\varphi) = \int_\Sigma \varphi^*(g_{ij})$$

$$\int_{\mathrm{Maps}(\Sigma,M)} e^{-S(\varphi)} \boxed{\cdots}\, d\varphi$$

things we want to measure

Renormalization: (clasically) governed by Ricci flow.
(remark: renormalization actually a semi-group)

dim $M = 2$ (Hamiton, mid 80s)

Curvature (Tangent vectors) $\otimes$ (Tangent vectors) $\to$ Tangent vectors

$$\nabla_X Y = Z$$
$$\parallel$$
$$\nabla_Y X$$

$$R(X,Y) = \nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - \nabla_{[X,Y]}$$

$$Rm(X, Y, Z, W) = \langle R(X,Y) Z, W \rangle$$

skew skew

Ric = contract the middle variables

$$Ric(e_i, e_j) = \sum_{k=1}^{\dim} Rm(e_i, e_k, e_k, e_j)$$

scalar curvature $R = \sum g^{ij} R_{ij}$

back to dim $M=2$ (Hamilton, mid 80s, B. Chow, ...)

If $g_{ij}(t)$ is evolving by Ricci flow, then R = scalar curvature satisfies

$$\frac{\partial}{\partial t} R = \Delta R + 2R^2$$ non-linear heat equation

dim $M=2$) $R_{ij} = R g_{ij}$

thm) Given a Riemannian metric on a compact surface, consider rescaled Ricci flow

$$g_{ij}(t)' = -2R_{ij} + \bar{R} g_{ij}$$

Exists for all time, and the $t \to \infty$ limit is a metric of constant curvature

Non-rescaled: $g=0$, vol $\to 0$ in finite time
$g=1$, vol $\to$ finite
$g>1$, vol $\to \infty$

(entire Ricci flow $g_{ij}(t) = e^{f(t)} g_{ij}(0)$)

classical thm) The conformal class of any metric on a compact surface has a unique constant curvature representative.

$M = X/\Gamma$, X = simply connected surface with a constant curvature metric. ie. $X = S^2, \mathbb{R}^2, \mathbb{H}^2$

$e(M) = -\quad 0 \quad +$

geometric: $X = H \backslash G$, $G = \mathrm{Isom}(X)$

3-manifolds (Thurston)

$M = X/\Gamma$ $\quad$ M is compact

$X = H \backslash G$ $\quad$ $G = \mathrm{Isom}(X)$, acting transitively

more generally, we can look at X/Γ s.t. $\mathrm{vol}(X/\Gamma) < \infty$

$\boxed{\text{8 geometries}}$ six of these geometries have circle ~~fibrations~~ Siefert fibrations

$$\begin{array}{c} S^1 \to M \\ \downarrow \\ \Sigma \end{array} \qquad \left(\begin{array}{l} z \mapsto e^{2\pi i k/n} z \\ \theta \mapsto \theta + \frac{2\pi}{n} \end{array} \right)$$

e = top. class of S^1-bundle

χ = euler number of base

	$\chi < 0$	$\chi = 0$	$\chi > 0$
$e = 0$	$S^2 \times \mathbb{R}$	$\mathbb{R}^3$	$\mathbb{H}^2 \times \mathbb{R}$
$e \neq 0$	S^3 (Hopf fibration)	Nil	$\widetilde{Sl_2}(\mathbb{R})$

2 geometries have T^2 "fibrations" over 1-dim. (either S^1 or interval)

Nil, Sol

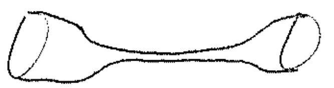
3 have constant curvature: S^3, $\mathbb{R}^3$, $\mathbb{H}^3$

Nil $\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$ $\qquad$ Sol $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$

(M, g_{ij}) , M compact, $\dim M = 3$

Let gradient flow commence

$S^3/\Gamma \to$ pt. in finite time

 → pinch off in finite time

$S^2 \times I = S^2 \times D^1$

$\partial(S^2 \times D^1) = S^2 \times S^0 = \partial(D^3 \times S^0)$

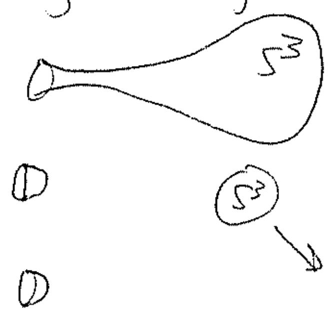
before the pinch-off occurs, do surgery, natural way so that Ricci flow continues

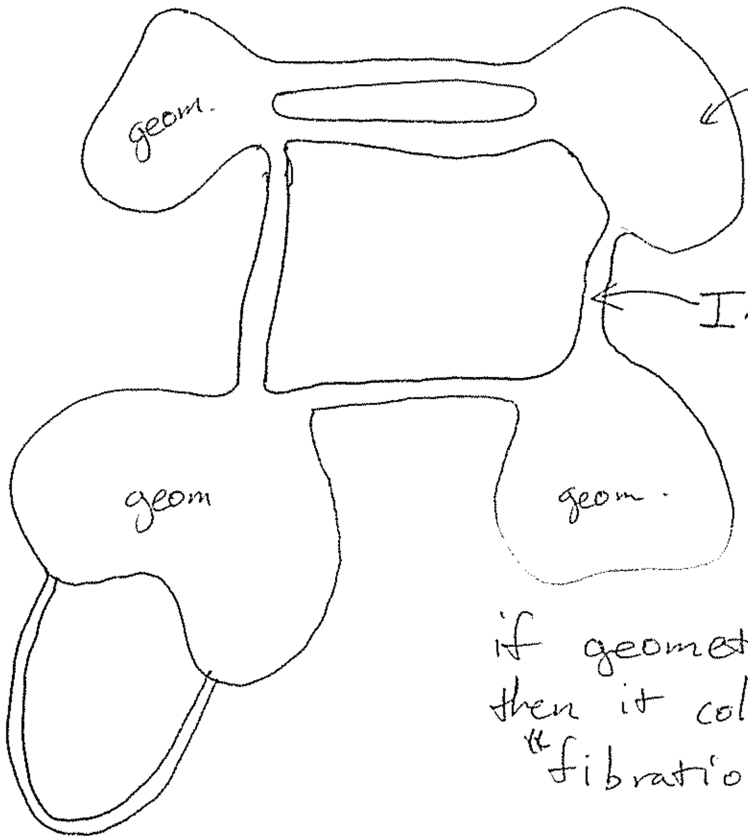
Perelman: – These are the only types of singularities in finite time
– No accumulation points of singularity times

note) Hypothesis: no embedded $\mathbb{RP}^2$s

Surgeries, extinctions happen at a discrete set of evolution times $t_1, t_2, t_3, \ldots$

for $t \gg 0$, only thing that happens is





"geometric" as $t \to \infty$

$I \times T^2$

geometric regions are either constant volume or growing

if geometric piece is <u>not</u> $\mathbb{H}^3/\Gamma$
then it collapses along S^1, T^2
"fibration" or T^3
seperate, not connected