Lecture by D.R. Morrison

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Ricci Flow, 3-Manifolds, & Physics (Part II)

<u>comment</u> $\mathcal{F} = \int_M (R + |\nabla f|^2) e^{-f} dV$

$n = \dim M$
f thought of as dilaton
τ a scale parameter

$$\mathcal{W}(g_{ij}, f, \tau) = \int_M \left[\tau(|\nabla f|^2 + R) + f - n\right] (4\pi\tau)^{-n/2} e^{-f} dV$$

restriction: $\int_M (4\pi\tau)^{-n/2} e^{-f} dV = 1$

$$\frac{\partial(g_{ij})}{\partial t} = -2R_{ij} \qquad \frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau} \qquad \frac{\partial \tau}{\partial t} = -1$$

<u>3 geometries</u> $(S^3, \mathbb{R}^3, H^3)$ $\qquad g_{ij}(t) = \lambda(t) g_{ij}(0)$ up to diffeo — <u>Ricci Solitons</u>

<u>5 geometries</u> split → growth, shrinking

So have stable solutions, could we have periodic behavior?

$$g_{ij}(t + \Delta t) = \lambda(t)\, g_{ij}(t)$$
up to diffeo

After introducing f, can show no periodic behavior.

<u>Aside</u> Intriguing remark "$\mathcal{W} = -$entropy," will get own lecture.

<u>Cosmology</u> spatially homogeneous?
G: acts transitively on spacelike hypersurfaces
$\mathfrak{g} = \mathrm{Lie}(G)$
Bianchi classified these I to IX (2 w/ continuous parameter)
why 9?

Partial answer: $X = H \backslash G$ $\quad G$ = isometry of X

find a subalgebra of $\mathfrak{g}$ of dim 3, acting (infinitesimally) transitively.

		dim n version
Bianchi I	$\longleftrightarrow$ flat, $X = \mathbb{R}^3$	$\mathbb{R}^n \rtimes SO(n)$
Bianchi V	$\longleftrightarrow$ hyperbolic, $X = \mathbb{H}^3$	$SO(n,1)$
Bianchi VI	$\longleftrightarrow$ sphere, $X = S^3$	$SO(n+1)$

isotropy groups of points for I, V, VI all $SO(n)$

Bianchi VIII $\longleftrightarrow$ $Sl_2(\mathbb{R})$

The Thurston Program

Step I: prime decomposition

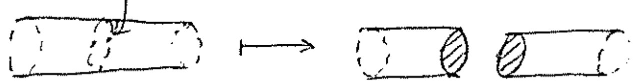
M = closed 3-manifold (eventually would like finite vol. Riem. 3-manifold with boundary)
$\underbrace{}_{\text{orientable}}$

$\boxed{\text{Kneser (1929)}}$ $S^2 \subseteq M$ essential, i.e. $S^2 \neq \partial B^3$, $B^3 \subseteq M$

construction: cut M along S^2, glue in B_1^3, B_2^3

2d analog

essential S^1



$\tilde{M}$ may be disconnected, or not. If $\not\exists$ essential S^2, call M prime

$\tilde{M} \ni p, q$ $\quad p \in B_1^3 \subseteq \tilde{M}$ $\quad q \in B_2^3 \subseteq \tilde{M}$

$\overline{B_2^3} \cap \overline{B_1^3} = \phi$

connected sum: $\hat{M} = \tilde{M} \setminus B_i^3$ $\quad \partial \overline{M} = S_1^2 \cup S_2^2$ $\quad$ glue in $S^2 \times [0,1]$

M

$\underline{\text{note}}$ S^3 is like 1

$\underline{\text{note}}$ $S^2 \times S^1$ is reducible, but prime

this is old history before Thurston

Step II consider $T^2 \hookrightarrow M$ s.t. $\pi_1(T^2) \hookrightarrow \pi_1(M)$

Klein $\hookrightarrow M$ $\pi_1(\text{klein}) \hookrightarrow \pi_1(M)$

Thurston's Geometrization Conjecture

Given M orientable, closed 3-manifold, prime

$\exists$ a collection of tori & Klein bottles

$(M - \cup T_i^2) = \coprod M_\alpha$ M_α is geometric

$M_\alpha = X/\Gamma$

boundary components diffeo to $T^2 \times [0, \infty)$

	Geometries	Cts. params.	non-compact quotients?
1)	S^3	No	No
2)	$\mathbb{R}^3$	Yes	No
3)	$\mathbb{H}^3$	No	Yes
4)	$S^2 \times \mathbb{R}$	Yes	No
5)	$\mathbb{H}^2 \times \mathbb{R}$	Yes	Yes
6)	$\widetilde{Sl_2(\mathbb{R})}$	Yes	Yes
7)	Nil	?	No
8)	Sol	Yes	No

1): get $\mathbb{RP}^3$, Lens spaces, $S^3/(\text{Binary icosahedron})$, etc.

2): get T^3/Γ_0, 10 of these

4): get $S^2 \times S^1$, $S^2 \times S^1/\mathbb{Z}_2 = \mathbb{RP}^3 \# \mathbb{RP}^3$

5): $S^1 \to M \downarrow \Sigma$ orbifold, compact, seifert fibration

5), 6): very similar structure

So arms come from $\mathbb{H}^3$ or Seifert circle bundles over orbifolds