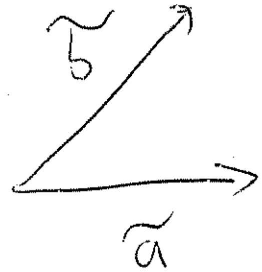
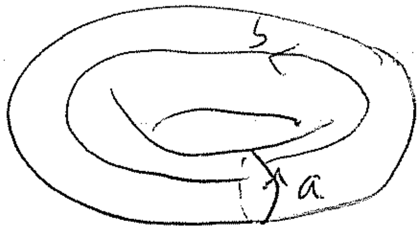
13 October 2006
D. Cooper

Aspects of the Topology and Geometry of 3-manifolds

Euclidean Torus $T = \mathbb{R}^2/(\mathbb{Z}\underset{\sim}{a} + \mathbb{Z}\underset{\sim}{b})$





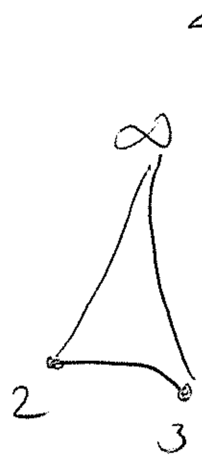
area = 1

<u>Moduli</u>

$\nearrow$ Teichmuller space $= \mathbb{H}^2 \cong \mathbb{R}^2$ (topologically)

$\searrow$ Moduli space = Mod surface

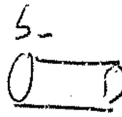
$= \mathbb{H}^2/SL(2,\mathbb{Z})$



Teichmuller $\longleftrightarrow$ <u>marking</u>

$$\left(\text{Automorphisms of } T\right) \longleftrightarrow GL(2,\mathbb{Z})$$

cut torus along geodesic b:



b_- b_+

length(b)

height = 1/length

glue back with a twist

~~Hyperbolic Surfaces~~

$$\mathbb{H}^2 = \begin{cases} \underline{\text{Parameters}} \\ \text{length} = \text{length}(b) \in (0,\infty) \\ \text{twist angle} \in (-\infty,\infty) \end{cases}$$

Hyperbolic surfaces

genus g, cut on $3g-3$

l_2 l_1 l_3

$\xrightarrow{\text{cut}}$ PANTS DECOMPOSITION

l_2 l_2 l_1 l_1 l_3 l_3

The lengths determine the pair of pants.

Twists for each pair.

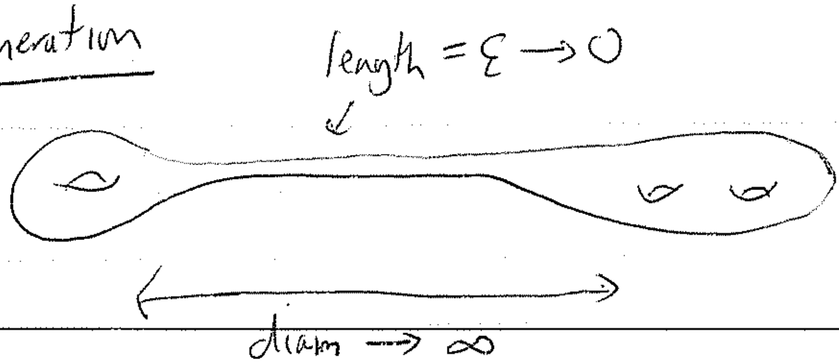
$$T(F_g) = \Big(\underbrace{(0,\infty)}_{\text{lengths}} \times \underbrace{(-\infty,\infty)}_{\text{twists}}\Big)^{3g-3} \cong \mathbb{R}^{6g-6} \text{ topologically}$$

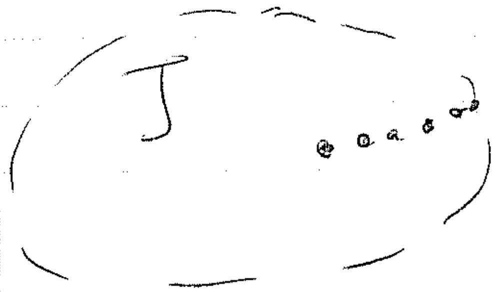
$$\cong \mathbb{R}^{|\chi|\cdot \dim(SL_2\mathbb{R})}$$

$$(\chi = 2-2g)$$

$$\cong \mathbb{R}^{|\chi|\cdot \dim(\mathrm{Isom}\,\mathbb{H}^2)}$$

Degeneration

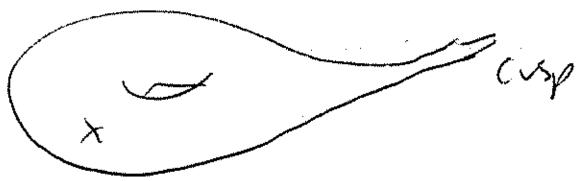




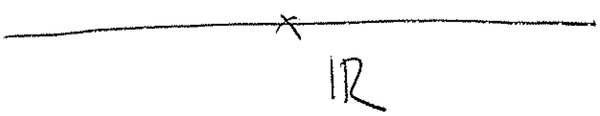
limit (Gromov-Hausdorff)

Choose basepoint on left, in middle, or on right

left.

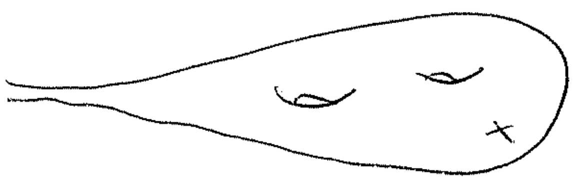


middle

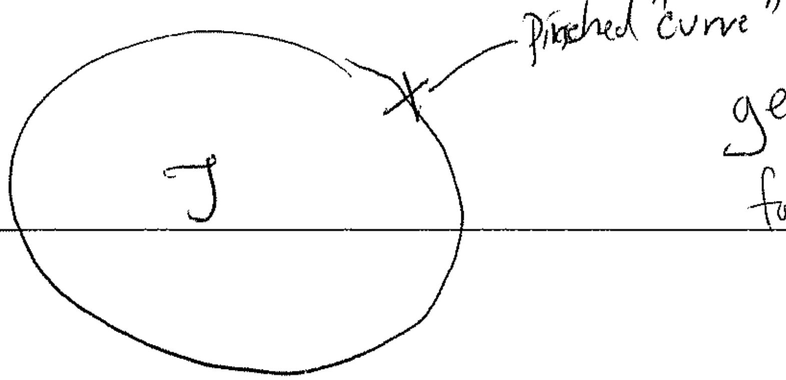


(real line)

right:

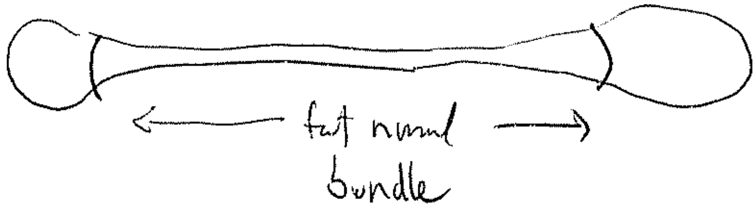


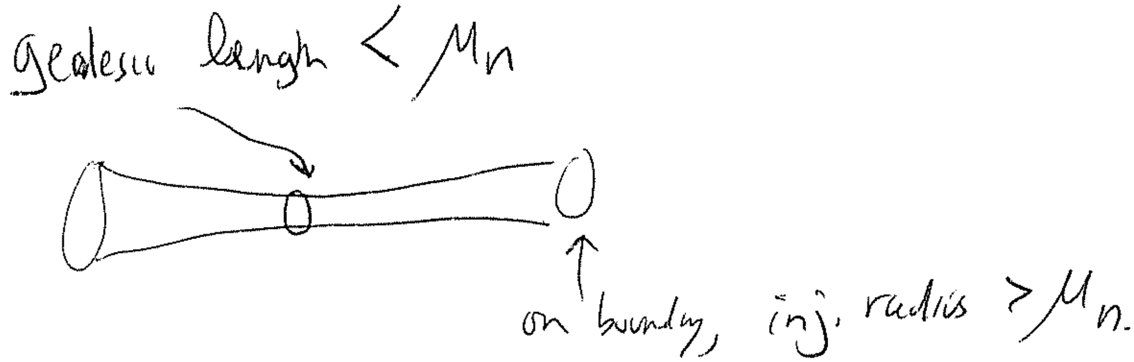
Thurston boundary



geodesic laminations form S^{6g-7}

<u>Margulis</u> in a hyp n-mfd, short geodesics have fat normal bundles.





<u>Thin parts are standard</u>

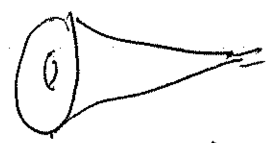
inj. radius $< \varepsilon$ (geodesic at point lgth $< 2\varepsilon$)

Γ = discrete $\subset \mathrm{Isom}\, \mathbb{H}^n$

$$[I+\varepsilon A, I+\varepsilon B] = I + O(\varepsilon^2)$$

(dim = 3: tubes $S^1 \times D^2$

cusps

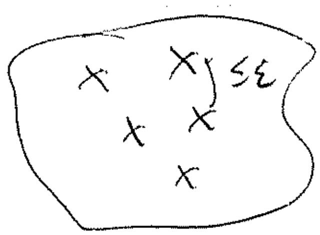


$T^2 \times [0, \infty)$

GH Topology

S = { all complete metric spaces w/ uniformly totally bounded }

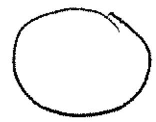
ε-net



$\forall \varepsilon\ \exists N$ s.t. $\forall X \in S\ \exists$ ε-net of X with $\leq N$ points.

<u>Thm</u> S is compact.

Limits are unique.

$(\varepsilon S^1)\times S^1 \xrightarrow[\varepsilon \to 0]{}$  $= S^1$

<u>collapse</u>

<u>Non-compact</u>, Introduce a basepoint.

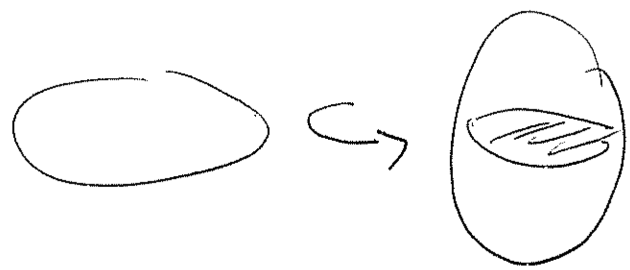
limit depends on basepoint

<u>Hyperbolic manifolds in dim ≥ 3.</u>

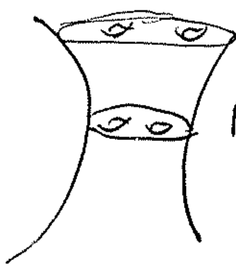
Mostow–Prasad rigidity

If M^n is hyperbolic, $n \geq 3$, $vol(M) < \infty$ <u>and</u> M is complete, then hyp. structure is unique.

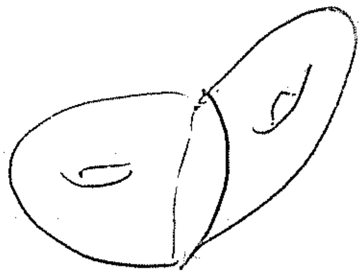
hyp surface $F = \mathbb{H}^2/\Gamma$

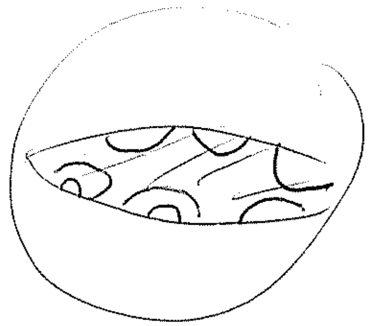
$\Gamma \subset \mathbb{H}^2 =$  $\mathbb{H}^3$

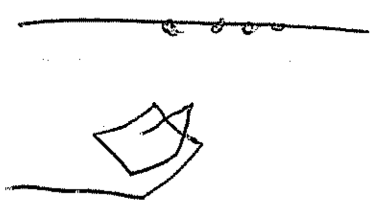
$F \times \mathbb{R} \underset{top.}{\cong} \mathbb{H}^3/\Gamma$



bending





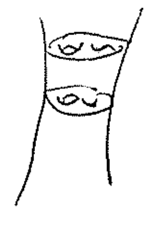


$\Gamma_{\text{bent}} \curvearrowright$ bent surface

Quasi Fuchsian (QF) 3-manifold

topologically $\cong$ $F \times \mathbb{R}$

geometrically

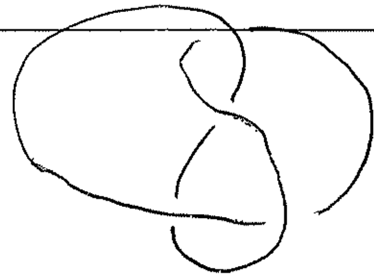


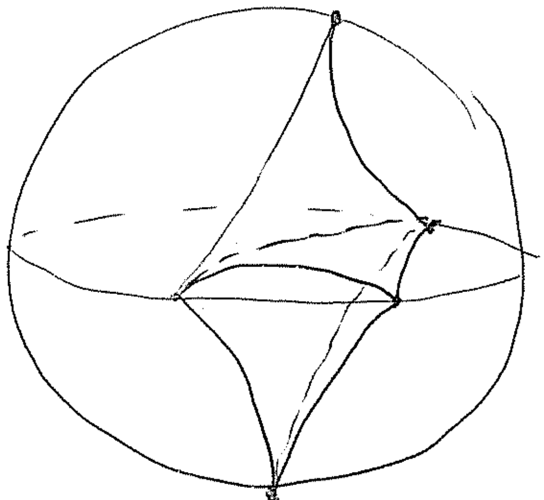
$$QF \sim \mathbb{R}^{|\chi| \cdot \dim(SL(2,\mathbb{C}))}$$

$\dim(SL(2,\mathbb{C})) = 6$

Hyperbolic Dehn Filling

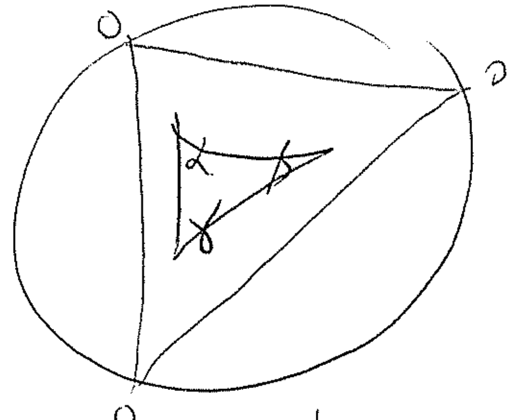
~~Figure~~ 8 Knot complement
trefoil

$S^3 -$  $= M_\infty$



two regular ideal tetrahedra

Ideal tetrahedra have a parameter, but unique regular one

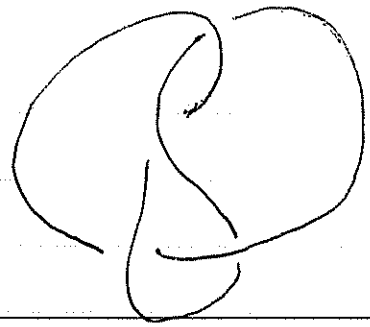


unique ideal hyperbolic triangle.

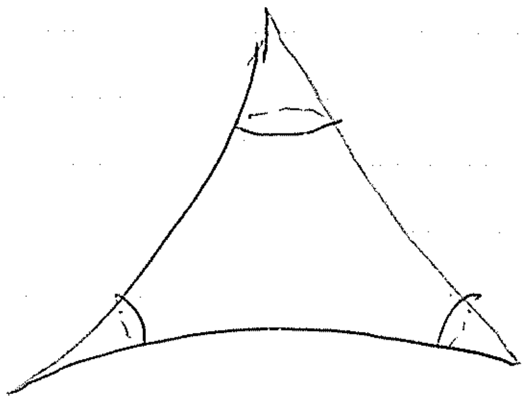
Pair faces to get M_∞.

volume $= 2 + \varepsilon$.

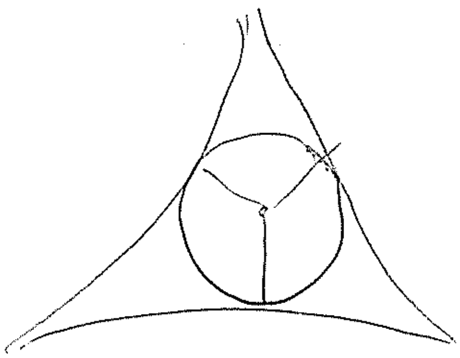
Figure 8 knot.



Incomplete structures

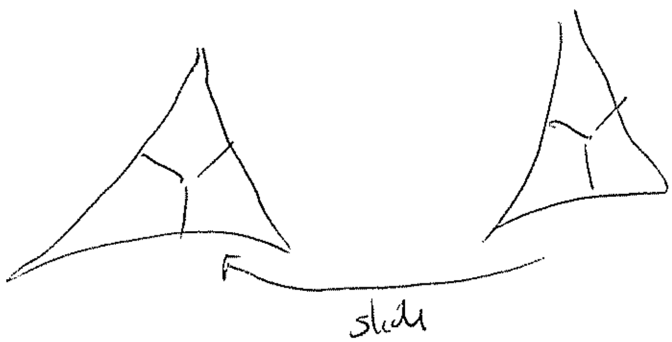


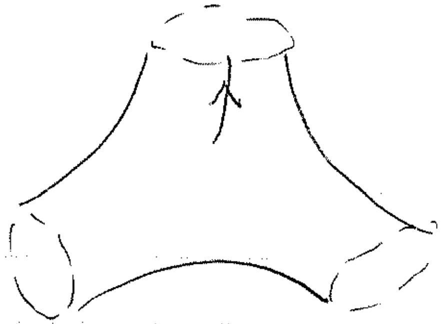
unique complete hyp. structure on $S^2 - \{\text{three points}\}$



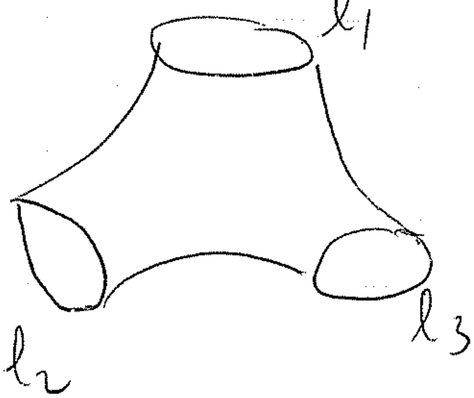
glue marked points.

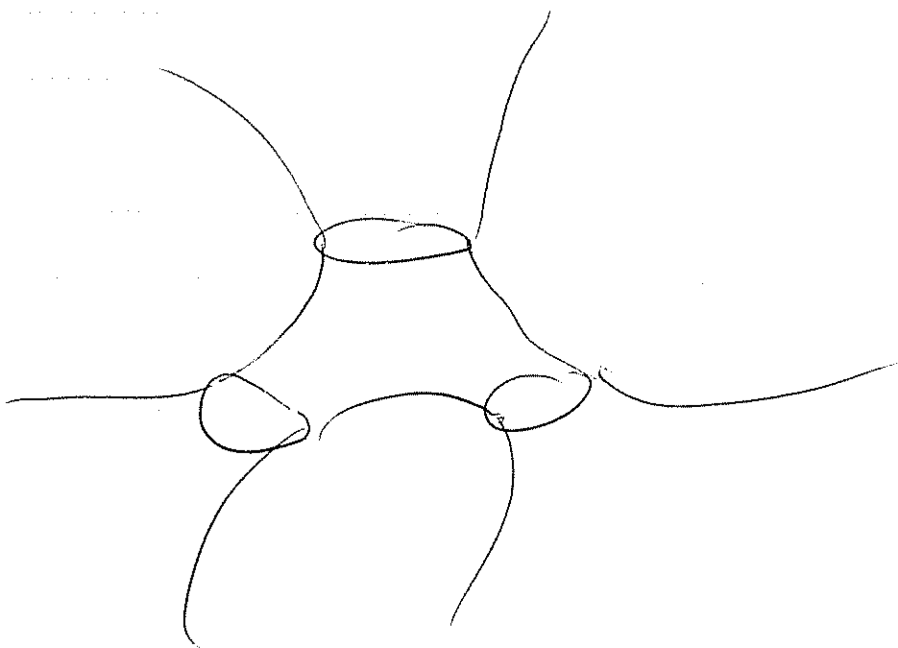
slide

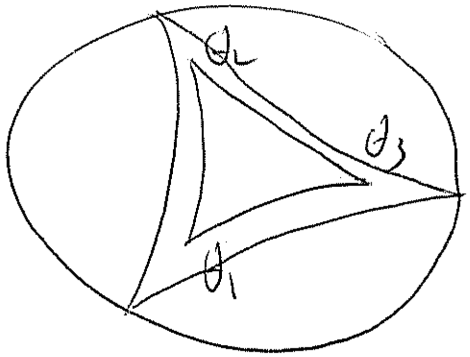


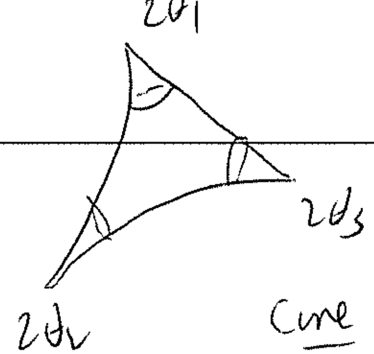


Compactify









cone metric

$$2\theta_i = \frac{2\pi}{n_i} \longleftrightarrow \text{discrete}$$

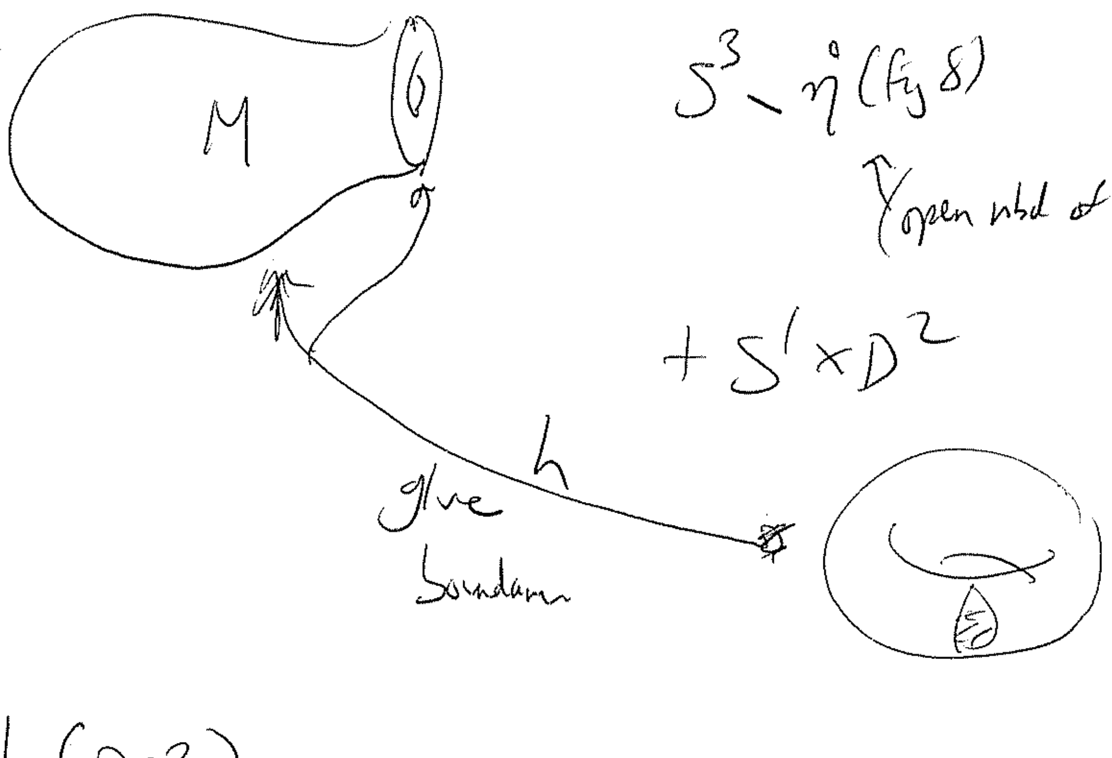
Dehn filling

$M = 3$-manifold, $\partial M = T$

$S^3 \setminus \eta$ (fig 8)

(open nbd of knot)

$+ S^1 \times D^2$

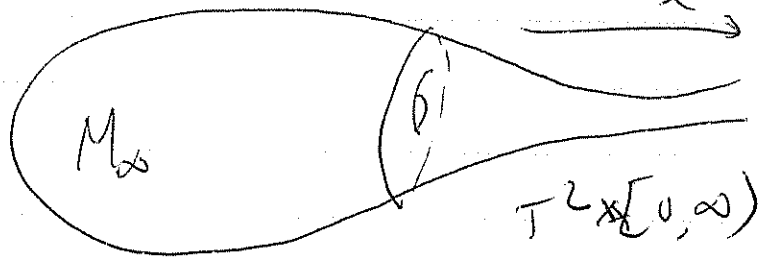


$h(\partial D^2)$

Dehn fillings $\longleftrightarrow$ Simple closed curve on T

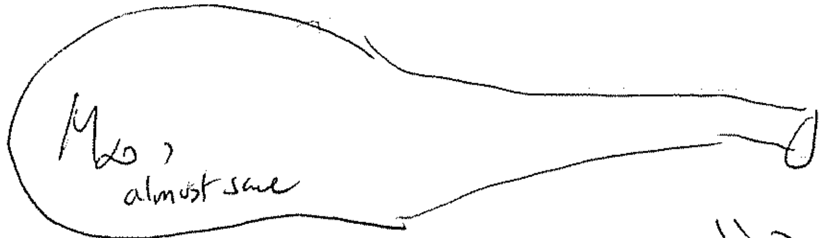
$\longleftrightarrow$ $m, n \in \mathbb{Z}$

$\longleftrightarrow$ $m/n \in \mathbb{Q} \cup \infty$



$T^2 \times [0,\infty)$, shrinking $e^{-\ell}$

$\rightsquigarrow$ incomplete structure

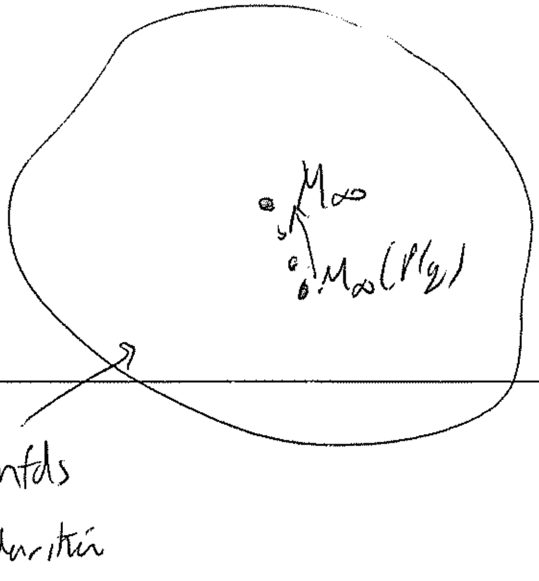


short geodesic

"Dehn filling".

Thurston's hyperbolic Dehn Surgery theorem

All but finitely many Dehn fillings on a cusp of a complete finite volume hyperbolic 3-manifold are hyperbolic and come from deformations



$|p| + |q| \to \infty$

Lickorish–Wallace

Every ~~compact~~ closed orientable 3-manifold is obtained by Dehn filling a link.

"Most hyperbolic"

classification

Produce a complete list without repetition.

1. List links.
2. List Dehn fillings
3. Compute hyp. structure or show $\nexists$ one
4. Check if one is already on list.