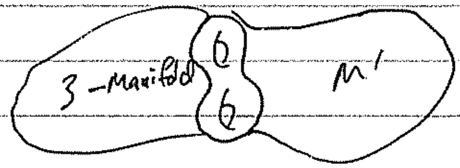
<u>Mike Freedman</u>

"Universal Manifold Pairings & TQFT's"

<u>Scribe</u> R. Eager

Universal Manifold Pairing $\sim$ TQFT

specialize to $2+1$



Surface $= S$

$\mathcal{M}_S$ - finite $\mathbb{C}$-linear combinations of manifolds M which bound S

$$\mathcal{M}_S \times \mathcal{M}_S \to \mathcal{M} \quad \leftarrow \text{closed 3-manifold}$$

$$\sum a_i M_i \times \sum b_j N_j \to a_i \bar{b}_j M_i \bar{N}_j \qquad \bar{N}_j \text{ - reverse orientation}$$

$$\begin{tikzcd}
\mathcal{M}_S \times \mathcal{M}_S \arrow[r] \arrow[d, "\text{kill } Z\text{-singular vectors}"] & \mathcal{M} \arrow[d, "Z = \int \mathcal{D}a \, e^{2\pi i k CS(a)}"] \\
V(S) \times V(S) \arrow[r] & \mathbb{C}
\end{tikzcd}$$

Chern-Simons Hilbert space

1989 Witten

Given a vector

$v \in \mathcal{M}_S \times \mathcal{M}_S$

call v $\begin{cases} \underline{\text{singular}} & \text{if } \langle v, w \rangle_\rho = 0 \quad \forall w \\ \underline{\text{null}} & \text{if } \langle v, v \rangle = 0 \end{cases}$

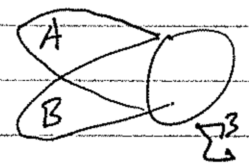
<u>Th^m</u> (Walker, Calegeri)

For any surface ρ_S is "positive" i.e. $\langle v, v \rangle_\rho = 0 \Rightarrow v = 0$

3+1 dim

Σ^3 has null vectors

$V = A - B$



Nothing is lost at universal stage

* Kevin's New Notes posted on Arxiv

Vaughn Jones – Diagonal Dominance

$\dot{\mathcal{M}}$ – set of 3mflds
$\mathcal{M}$ – vector space
$\mathcal{O}$ – ordered set

Come up with a complexity:

$c : \dot{\mathcal{M}} \longrightarrow \mathcal{O}$

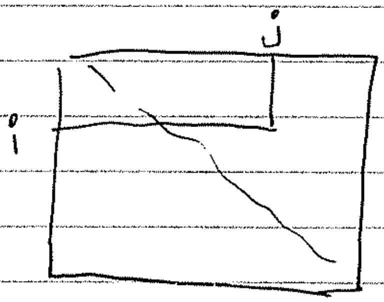
$\partial M_i = S = \partial M_j$

$c(M_i M_j) < \mathrm{Max}(c(M_i M_i), c(M_j M_j))$

gluing right mfld w/ orientation reversed

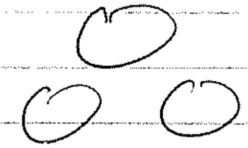
$\Rightarrow$ Thm $\quad \langle \sum a_i M_i, \sum a_i M_i \rangle = \sum_{ij} a_i \bar{a}_j M_i M_j$

real + positive on diagonal

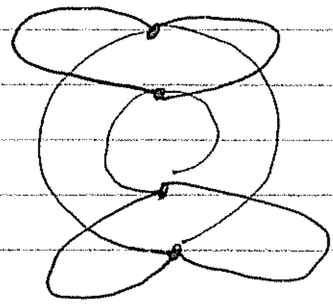


Warm-up Example

1- manifold



$c(\text{1-manifold}) = \#$ components



$$c(M_iM_j) < \mathrm{Max}\Big(\underset{\underset{2}{\|}}{cM_iM_i},\ \underset{\underset{2}{\|}}{cM_jM_j}\Big)$$

3-manifold complexity function:

$c := c^3$

c_0, c_1

$c^1 = c_0 \times c_1$ lexicographical order

$c^2 = c_0 \times c_1 \times c_2 = c^1 \times c_2$

$c^3 = c_0 \times c_1 \times c_2 \times c_3$

$c_3 = c_{\text{Siefert fibered}} \times c_{\text{hyperbolic}} \times c_i$

χ

$(-\text{hyperbolic volume}, -\text{real length spectrum})$

Ignore rotations
loxodromy

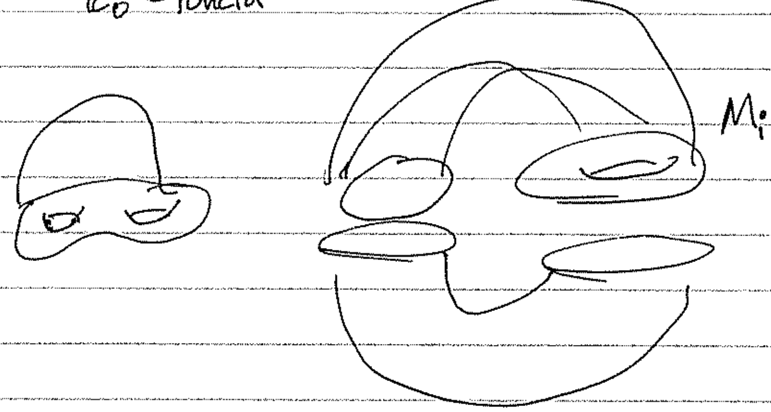
For c^k either

$c^k(M_iM_j) < \mathrm{Max}(c^k(M_iM_i), c^k(M_jM_j))$ (diagonal dominance)

or equality holds and

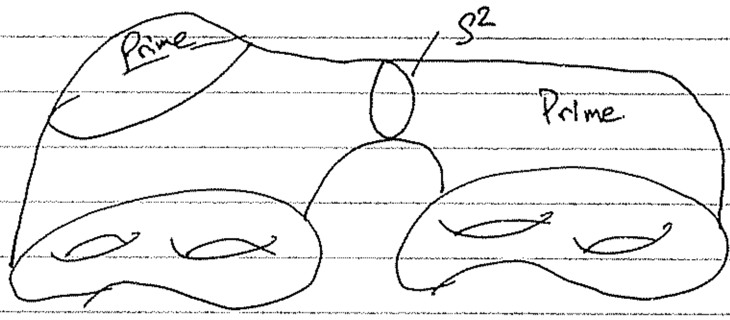
(M_i, S) and (M_j, S) are k-similar

0-similar (component level)
" π_0 - functor "



1-similar Finite group TQFT
π_1 Kernels identical

2-similar
location of 2-spheres Like Dehn's Lemma



blueprints agree if you ignore the primes

3-similar
Primes, and Seifert fibration info

Homotopy theory captures most of manifold topology up to Reideimester torsion
Can mostly be detected by finite group TQFT

3-similar case

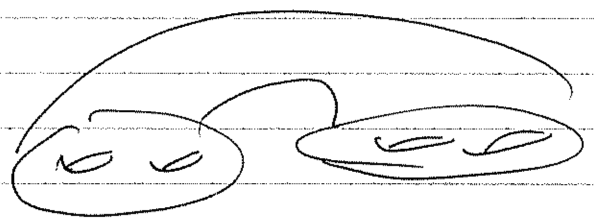
$S \subset M$ $S^2, T^2 \not\subset S$

where $\boxed{S = \partial M}$

* $\pi_1(S) \xhookrightarrow{\text{injects}} \pi_1(M)$
* No essential annuli
* $\pi_2(M)=0$ $\nexists\ \mathbb{Z}\oplus\mathbb{Z} \subset \pi_1(M)$

$\exists!$ a hyperbolic metric on M (finite volume)
s.t. S is totally geodesic

Generally hyperbolic structures have a large moduli space



$\Rightarrow$ Volume(M) & length spectrum
are topological invariants

$\mathcal{M}_S^{hyp} \times \mathcal{M}_S^{hyp} \to \mathcal{M}^{hyp}$

M_iM_j also has a unique hyperbolic metric

{ Totally geodesic: 2 pts on mfld

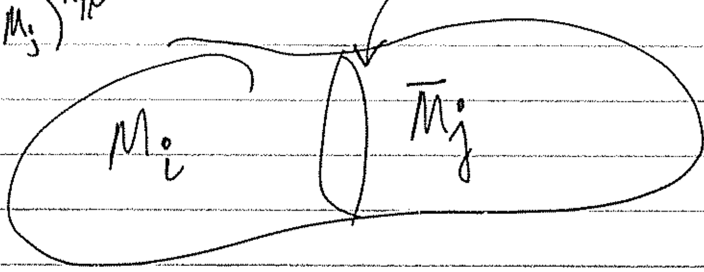
Thm ρ^{hyp} is positive

Pf Diagonal Dominance w.r.t. $C_{hyp} = (-\text{Vol} - \text{real length spectrum})$

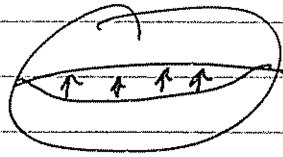
Lemma Vol M_iM_j < Max(Vol M_iM_i, Vol M_jM_j)
(Agol, Storm, Thurston)
"Actually this conjecture is in Agol's thesis
He was my PhD student, but I didn't read it"

$(M_i M_j)^{hyp}$

S stable minimal surface



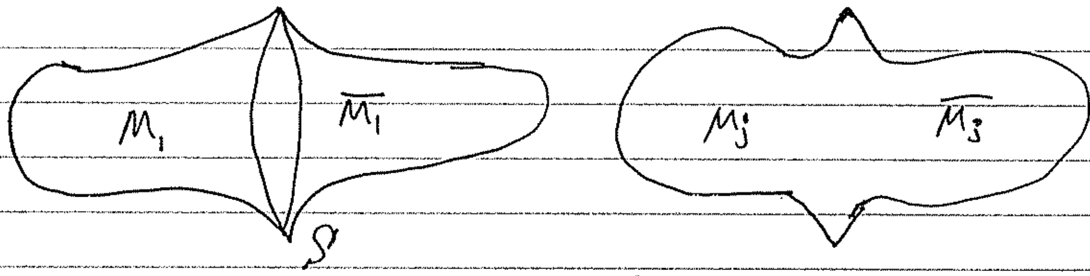
Equator is an unstable geodesic



decrease length

Brilliant Crazy Idea

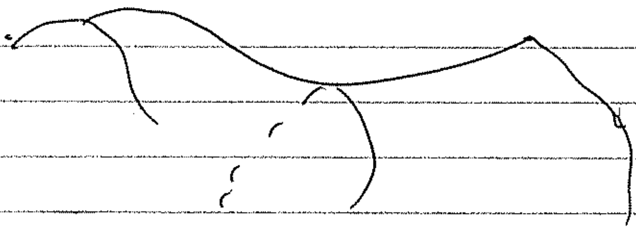
double surface along S despite not being totally geodesic



in spite of the singular Riemannian metrics

Scalar curvature have $R=-6$ everywhere

$$B_t(p) = \frac{4}{3}\pi\left(1 - \frac{R(p)}{30}t^2 + O(t^3)\right)t^3$$



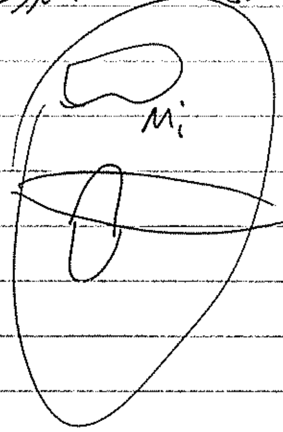
Minimal Surface

Ricci flow – Normalized volume decreases as you flow to $(M_i M_i)^{hyp}$

Equality occurs in case of isometric gluing

$\Rightarrow$ induces same hyperbolic structure on S

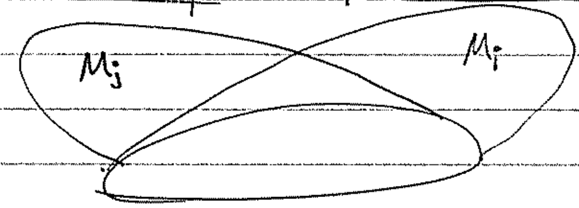
Essential Length Spectrum



$M_i = M_j$

essential arc

1-similarity   Finite Group TQFT



$$K_i \to \pi_1(S) \to \pi_1(M_i)$$
$$K_j \to \pi_1(S) \to \pi_1(M_j)$$

$\exists\, x \in K_i \setminus K_j$ show there exists a finite group F such that

$$Z_F(M_i; M_j) < \mathrm{Max}\big(Z_F(M_i; M_i),\ Z_F(M_j; M_j)\big)$$

Residual Finiteness   f.g. matrix groups over $\mathbb{C}$ residually finite

$$\begin{tikzcd}
 & \pi_1(M_i) \arrow[dr, dashed] & \\
\pi_1(S) \arrow[ur] \arrow[dr] & 1 & F \\
x \arrow[ur, mapsto] \arrow[r, mapsto] & y \neq 1 \arrow[r, mapsto] & w \neq 1 \\
 & \pi_1(M_j) \arrow[uur] &
\end{tikzcd}$$

Geometrically, F-principal bundle

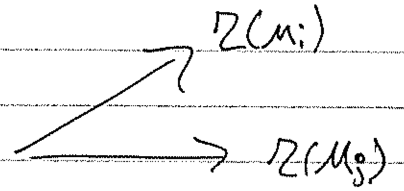
(1.) $V_F(S)$

$\Downarrow$

$Z(M_i) \neq Z(M_j)$

(2.) V_F is a euclidean TQFT

$Z(M_i M_j) = \langle Z(M_i), Z(M_j), \text{etc}\ldots)$



Cauchy-Schwartz — max occurs on diagonal

$\therefore$ Diagonal Dominance

$Z(M_i M_j) < Max(Z(M_i M_i), Z(M_j M_j))$

We need to look at all finite groups

either $\nexists$ Z distinguishes M_i and M_j

or $\nexists$ $\ker \pi_1(S) \to \pi_1(M_i)$

$= \ker \pi_1(S) \to \pi_1(M_j)$