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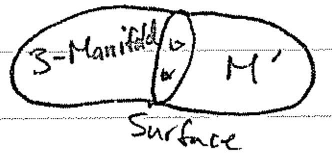
notes by: M. Gary

Universal Manifold Pairings II

Universal manifold pairings ~ TQFTs
2+1 dim

this week



$\mathcal{M}_S$ = finite linear comb. of M st $\partial M = S$

$$\mathcal{M}_S \times \mathcal{M}_S \longrightarrow \mathcal{M}$$ closed 3 manifold (or formal linear combo)

$$\left(\sum a_i M_i\right) \times \left(\sum b_j N_j\right) \longmapsto a_i \bar{b}_j M_i \bar{N}_j$$

$$\begin{array}{ccc} \mathcal{M}_S \times \mathcal{M}_S & \xrightarrow{\rho} & \mathcal{M} \\ & & \Big\downarrow Z \left(= \int \mathcal{D}a \, e^{-2\pi i k \, C.S.(a)}\right) \\ & & \mathbb{C} \end{array}$$

~~1989 Witten (Jones Poly.)~~

def) v is singular if $\langle v, w \rangle_\rho = 0 \ \forall w$

def) v is null (light-like) if $\langle v, v \rangle_\rho = 0$

thm) (Walker, Calagari) $\forall$ surface S, ρ_S is "positive"
i.e. $\langle v, v \rangle_{\rho_S} = 0 \Rightarrow v = 0$

$$\begin{array}{ccccc} \mathcal{M}_S & \times & \mathcal{M}_S & \xrightarrow{\rho} & \mathcal{M} \\ \Big\downarrow & \text{quotient by } \ker(Z) & \Big\downarrow & & \Big\downarrow Z \\ \boxed{V(S)} & \boxed{\times} & \boxed{V(S)} & \boxed{\longrightarrow} & \boxed{\mathbb{C}} \end{array}$$

See Kevin's notes on Arxiv

Physicists are used to this

Diagonal Dominance

Come up with a complexity

$c : \mathring{\mathcal{M}} \longrightarrow \mathcal{O}$ s.t.

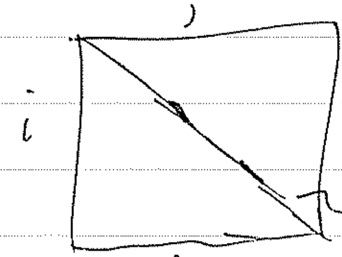
set of 3-manifolds — ordered set

$\partial M_i = S = \partial M_j$

$c(M_i M_j) \underset{\uparrow \text{strict}}{<} \max\left(c(M_i, M_i), c(M_j, M_j)\right)$

$\Rightarrow$ thm.

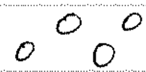
$\langle \sum a_i M_i, \sum a_j M_j \rangle = \sum_{i,j} \underbrace{a_i \bar{a}_j}_{\text{real \& pos. on diagonal}} M_i M_j$



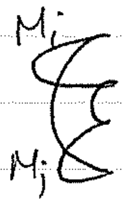
complexity maximal on diagonal

if $c \neq$, then manifolds $\neq$

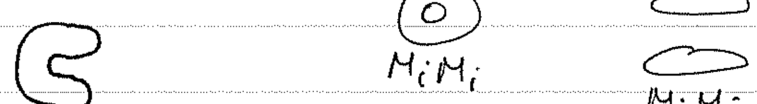
eg) 1-manifolds :



let $c(\text{1-manifold}) = \#$ components



$$c(M_i M_j) < \max\left(c(M_i M_i), c(M_j M_j)\right)$$
$$\quad\; \overset{\parallel}{1} \qquad\qquad\qquad\quad \overset{\parallel}{2} \qquad\qquad \overset{\parallel}{2}$$



$M_i M_j$ $\qquad$ $M_i M_i$ $\qquad$ $M_j M_j$

coming up with a complexity function in $d=3$ is much harder.

$C := C^3$

C_0, then C_1

$C^1 = C_0 \times C_1$ lexicographically (ie. C_0 more important than C_1)

$C^2 = C_0 \times C_1 \times C_2 = C^1 \times C_2$

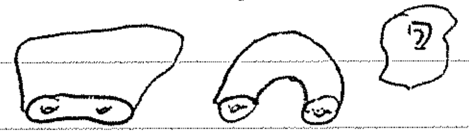
$C^3 = C_0 \times C_1 \times C_2 \times C_3$

$C_3 = C_{\text{Sieft. Fibered}} \times C_{\text{hyperbolic}} \times C_i$

$\chi_Q[X]$ (hyperbolic vol, real length spectrum

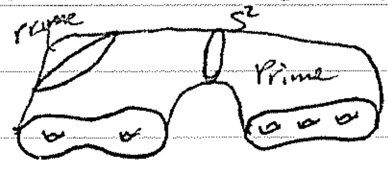
for C^k, either have diagonal dominance
or M_i and M_j as pairs,
(M_i, S) and (M_j, S) are <u>k-similar</u>,
and $C^k(M_i M_j) = C^k(M_i M_i) = C^k(M_j M_j)$

<u>0-similar</u>: "π_0 - functor"



<u>1-similar</u>: π_1 kernels identical
Finite-group TQFT stuff is here

<u>2-similar</u>: location of 2-spheres
the only thing unknown now is the nature of the primes in the decomposition



<u>3-similar</u>: deals with the primes

Consider $S = \partial M$, $S^2, T^2 \notin S$, no essential annuli,
$\pi_1(S) \xrightarrow{\text{injects}} \pi_1(M)$, $\pi_2(M) = 0$, $\nexists\, \mathbb{Z} \oplus \mathbb{Z} \subset \pi_1(M)$
call this $\mathcal{M}_S^{hyp}$

<u>Thurston</u> $\exists!$ hyperbolic metric on M (finite volume)
s.t. S is totally geodesic

$\Rightarrow$ volume(M) is an invariant, similarly w/ lengths

$M_i M_j$ also has a ! hyperbolic metric.

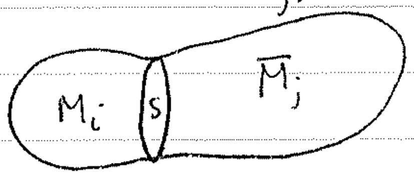
so form $\mathcal{M}_S^{hyp} \times \mathcal{M}_S^{hyp} \xrightarrow{\rho^{hyp}} \mathcal{M}^{hyp}$

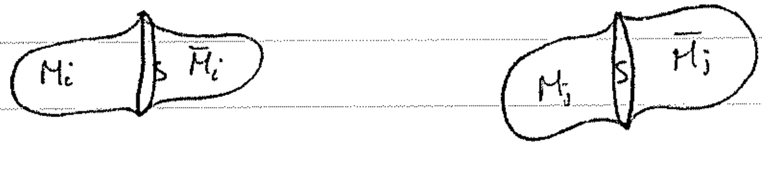
<u>note</u> totally geodesic: $x, y \in S$,
$\gamma : I \to S$, $\gamma(0) = x$, $\gamma(1) = y$, γ a geodesic
length$(\gamma) \leq$ length(ζ) $\forall\ \zeta$ geodesics
connecting x & y in M

<u>thm</u> ρ^{hyp} is positive
<u>pf</u> DD WRT $c_n = (-\text{vol}, -\text{real length spec.})$

<u>lemma</u> $\text{vol } M_i M_j \geq \min(\text{vol } M_i M_i, \text{vol } M_j M_j)$
<u>pf</u> (Agol, Storm, Thurston)

hyperbolize $(M_i M_j)^{hyp}$ S can be made a stable minimal surface





despite the singularity, the Sing. Riemannian metrics have $R = -6$ everywhere

<u>aside</u> How to define R for a singular manifold

$$B_t(p) = \frac{4\pi}{3} t^3 \left(1 - \frac{R(p)}{30} t^2 + O(t^3)\right)$$

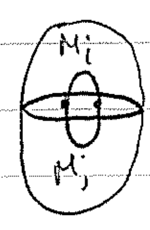
up to 6^{th} order, minimal surfaces are democratic in dividing volume

Use a Lemma by Perelman & Elliptic PDE
drop both $M_i M_i$ & $M_j M_j$ into Ricci flow

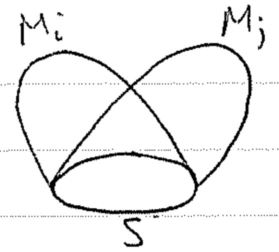
as go to $(M_i M_i)^{hyp}$, vol. decreases.
@ end of Ricci flow, vol is $2\,\text{vol}\, M_i + 2\,\text{vol}\, M_j$ ✓

So either conclude diagonal dominance or equality. In case of equality, induce same hyperbolic geometry on S.

<u>Length Spectrum</u>



Shortest new arc comes from gluing arcs together. if $M_i \neq M_j$, this new geodesic will be longer than in $M_i M_i$ or $M_j M_j$



$$K_i \longrightarrow \pi_1(S) \longrightarrow \pi_1(M_i)$$
$$K_j \longrightarrow \pi_1(S) \longrightarrow \pi_1(M_j)$$

$\exists\, x \in K_i \setminus K_j$ show $\exists$ finite group F s.t.
$Z_F(M_i M_j) < \max\left(Z_F(M_i M_i),\ Z_F(M_j M_j)\right)$

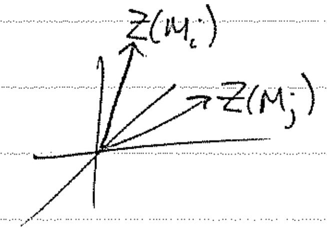
$$\begin{tikzcd}
 & \pi_1(M_i) \arrow[dr, dashed, "\text{cannot be extended}"] & & \\
\pi_1(S) \arrow[ur, "x \mapsto 1"] \arrow[r] & \pi_1(M_j) \arrow[r] & F & \text{finite group}
\end{tikzcd}$$

Residual finiteness:

$$x \longmapsto y \neq 1 \longmapsto w \neq 1$$

So F distinguishes M_i and M_j

① $Z(M_i) \neq Z(M_j) \in V_F(S)$
② V_F is a ~~unitary~~ actually Euclidean TQFT



$Z(M_i M_j) = \langle Z(M_i), Z(M_j) \rangle$, etc.

but now Cauchy-Schwarz $\Rightarrow$ Diagonal Dominance
$Z(M_i M_j) < \max\left(Z(M_i M_i),\ Z(M_j M_j)\right)$

Since can list all finite groups F in order, just use Z_F in lexigraphical order.