10 November 2006
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Review

How much topology do unitary TQFTs see?

[A] $n \geq 4$ $\exists\, M^n, N^n$, $\partial M = \partial N = Y^{n-1}$ $M \not\cong N$ such that

$Z(M) = Z(N) \in V(Y)$ for all UTQFTs (Z, V).

[B] $n = 4$ If M^4, N^4 are closed, simply connected and homotopy equivalent, then $Z(M) = Z(N) \in \mathbb{C}$ $\forall$ UTQFTs Z.

[C] $n \leq 2$, if $Z(M^2) = Z(N^2)$ $\forall\, Z$
$\Leftrightarrow (M, Y) \cong (N, Y)$.

[D] $n = 3$ "Z_{univ}" $(M^3) = Z_{univ}(N^3) \Leftrightarrow (M, Y) \cong (N, Y)$

$\boxed{\text{Does [C] hold for } n = 3\text{?}}$

Finite Group TQFTs
(untwisted)

(Dijkgraaf - Witten
Freed - Quinn
CMP '91)

G = finite group

$P_G(X^k) = \left\{ \begin{matrix} G \to P \\ \downarrow \\ X \end{matrix} \right\}$ up to isomorphism

$$Z_G(M^3_{\text{closed}}) = \sum_{\rho \in \mathcal{P}_G(M)} \frac{1}{|\mathrm{Aut}(\rho)|} = \frac{1}{|G|} \cdot |\{\pi_1(M) \to G\}|$$

$$|\text{orbit } \rho| \cdot |\text{Stab}(\rho)| = |G|$$

$$\partial M = \emptyset, \quad \begin{matrix} G \to Q \\ \downarrow \\ \partial M \end{matrix} \qquad Q \in \mathcal{P}_G(\partial M)$$

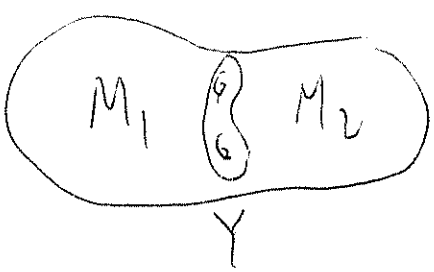
$$Z(M, Q) = \sum_{\substack{\rho \in \mathcal{P}(M) \\ \rho|_{\partial M} = Q}} \frac{1}{|\mathrm{Aut}(\rho)|} = |\{\rho \in \mathcal{P}(M) \mid \rho|_{\partial M} = Q\}|$$

$$Z(M) : \mathcal{P}_G(\partial M) \to \mathbb{C}.$$

$$V(Y^2) = Z(Y^2) = \{\mathcal{P}_G(Y) \to \mathbb{C}\}$$

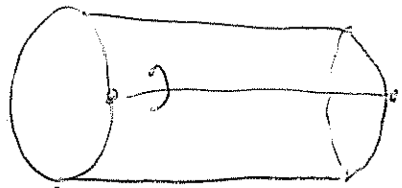
$$\langle \delta Q_1, \delta Q_2 \rangle = \begin{cases} 0, & Q_1 \not\cong Q_2 \\ \frac{1}{|\mathrm{Aut}\, Q|}, & Q_1 \cong Q_2 \end{cases}$$

$$Z(M^3) \in Z(\partial M).$$



$$Z(M_1 \cup_Y M_2) = \langle Z(M_1), Z(M_2) \rangle_{Z(Y)}$$

$M = S^2 \times I$



S^2 $\quad S^1 \times I$

$g \in G$.

$P_G(X) \longleftrightarrow \{X \longrightarrow BG\}/\text{homotopy}$

$$\begin{tikzcd} G \arrow[r] & EG \arrow[d] \\ & BG \end{tikzcd}$$

EG is contractible

$EZ_2 = S^\infty \subseteq \mathbb{R}^\infty$

$BZ_2 = \mathbb{R}P^\infty$

$BU(1) = \mathbb{C}P^\infty$

$BSU(2) = \mathbb{H}P^\infty$

"fields" $\mathcal{C}(X^k) = \{X \to BG\}$

local relation: $f_1 \sim f_2$ iff f_1 is homotopic to f_2

$A(Y^2; c)$

$\mathcal{C}(X) \xrightarrow{r} \mathcal{C}(\partial X)$

$c \in \mathcal{C}(\partial X)$

$\mathcal{C}(X; c) = r^{-1}(c)$

$\mathcal{C}(\emptyset) = pt$

$\mathcal{C}(X \sqcup Y) = \mathcal{C}(X) \times \mathcal{C}(Y)$

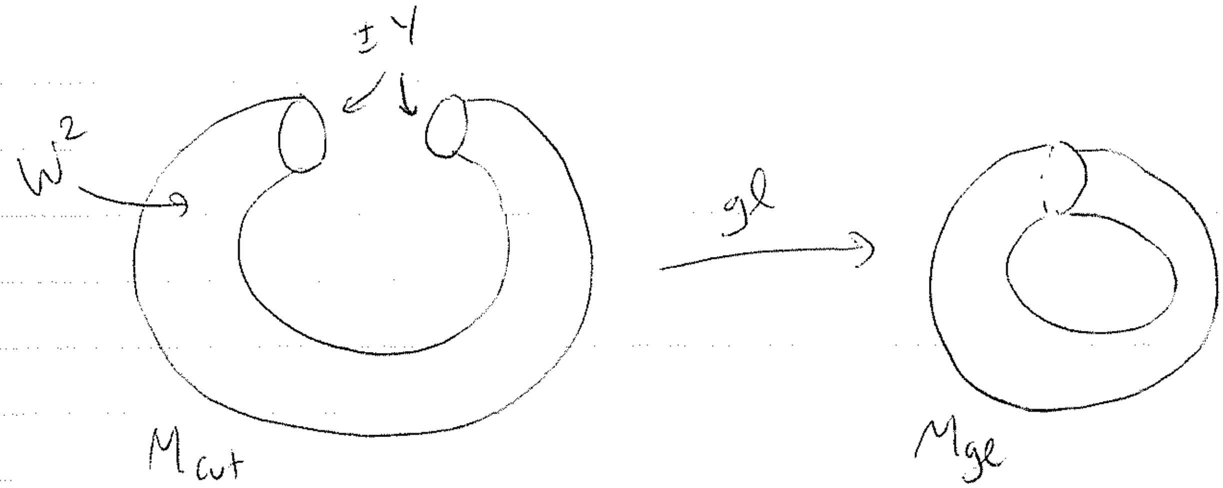
$$\begin{tikzcd}
 & \mathcal{C}(X \cup_S Y) \arrow[dl] \arrow[dr] & \\
\mathcal{C}(X) \arrow[dr, "r"'] & & \mathcal{C}(Y) \arrow[dl, "r"] \\
 & \mathcal{C}(S) &
\end{tikzcd}$$

$$A(Y^2; c) \overset{\text{def}}{=} \mathbb{C}[\mathcal{C}(Y; c)] / f_1 \sim f_2$$

$$Z(Y^2; c) \overset{\text{def}}{=} \{\varphi : \mathcal{C}(Y; c) \to \mathbb{C} \mid f_1 \sim f_2 \Rightarrow \varphi(f_1) = \varphi(f_2)\}$$

Want: $Z(M^3) \in Z(\partial M)$

$Z(Y) = A(Y)^*$



$\partial M_{cut} = W^2 \cup Y \cup -Y$ $\qquad$ $\partial M_{gl} = gl(W)$

$\partial Y = S, \ c \in \mathcal{C}(S)$ $\qquad$ $gl_c(f_{cut})$

$f_{cut} \in A(W; c, \bar{c})$

$Z(M_{gl}) : A(\partial M_{gl}) \longrightarrow \mathbb{C}.$

① $\boxed{Z(M_{gl})(gl_c(f_{cut})) = \sum_i \dfrac{Z(M_{cut})(f_{cut} \cup e_i \cup \bar{e_i})}{\langle e_i, e_i \rangle}}$

$\{e_i\}$ = orthog basis of $A(Y; c)$

Also Want: inner product on $A(Y^2, c)$

$\Rightarrow \quad x, y \in A(Y^2; c)$

② $\boxed{\langle x, y \rangle = Z(Y \times I)(\bar{x} \cup y)}$

$$Z(B^3) \in Z(S^2) \cong \mathbb{C}.$$

$$e_{S^2} = \text{constant map } S^2 \to * \in BG.$$

$$e_{S^2} \in A(S^2)$$

$$Z(B^3)(e_{S^2}) = \lambda \in \mathbb{C} \qquad \begin{matrix}\text{eventually}\\ (\lambda = 1).\end{matrix} \qquad \pi_i(BG) = \begin{cases} 0, & i \geq 2 \\ G, & i = 1 \\ 1, & i = 0 \end{cases}$$

$$Z(M) \in \underset{\overset{\shortparallel}{V(\partial M)}}{Z(\partial M)} = A(\partial M)^*$$

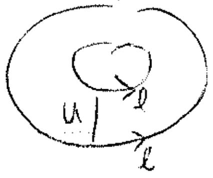
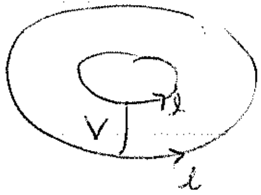
Use ② with $Y = D^2$:

$$e_{D^2} \in A(D^2; e)$$

$$\langle e_{D^2}, e_{D^2} \rangle = Z(\underset{\overset{\shortparallel}{B^3}}{D^2 \times I})(\underset{\overset{\shortparallel}{e_{S^2}}}{\bar{e}_{D^2} \cup e_{D^2}}) = \lambda$$

$$Z(S^1 \times D^2)(m, l) = \begin{cases} 0, & m \neq 1 \\ \lambda \cdot \frac{1}{\lambda} = 1, & m = 1. \end{cases}$$

$$Z(H_g)\underset{1 \leq i \leq g}{(m_i, l_i)} = \begin{cases} 0, & \text{some } m_i \neq 1 \\ \lambda^{1-g}, & \text{all } m_i = 1. \end{cases}$$

$$\Big\langle$$  ,  $$\Big\rangle = \begin{cases} 1, & u = v \\ 0, & u \neq v. \end{cases}$$

$[u, l] = 1$

$[v, l] = 1$

$$Z(S^2 \times I)(e_{S^2} \amalg e_{S^2}) = \lambda^2 \cdot |G|.$$

$$S^2 \times I = D^2 \times I \cup_{S^1 \times I} D^2 \times I$$

$$\langle e_{S^2}, e_{S^2} \rangle = Z(S^2 \times I)(\overline{e_{S^2}} \cup e_{S^2}) = \lambda^2 \cdot |G|.$$

$$Z(S^3) = Z(B^3 \cup B^3) = \lambda^2 \cdot (\lambda^2 |G|)^{-1} = \frac{1}{|G|}.$$

$$Z(B^3 \amalg B^3) \cdot \frac{1}{\langle e_{S^2}, e_{S^2} \rangle}$$

$$Z(S^1 \times S^2) = Z(S^2 \times I) \frac{1}{\langle e_{S^2}, e_{S^2} \rangle} = \lambda^2 |H| \cdot \frac{1}{\lambda^2 |H|} = 1.$$

$$\boxed{\lambda = 1}$$

Claim

M ~~is connected~~ has no closed components, $\partial M \neq \emptyset$ then $Z(M)(f) = \pi_0 \{ F : M \to BG \mid F|_{\partial M} = f \}$

$f : \partial M \to BG$

Pf: (a) true for $M_1, M_2 \Rightarrow$ true for $M_1 \amalg M_2$

(b) true for $M = B^3$

(c) true for $M \Rightarrow$ true for $M \cup 1h$

(d) " - - - - $M \cup 2h$

If M is closed, connected then

$$Z(M) = \frac{1}{|G|} |\mathrm{Hom}(\pi_1(M) \to G)|$$

$$= \sum_{p \in P_G(M)} \frac{1}{|\mathrm{Aut}(p)|}$$

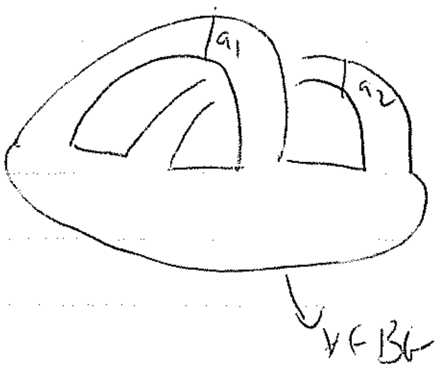
$$Z(M) = Z(M \setminus B^3)^{(e_{S^2})} Z(B^3)^{(e_{S^2})} \frac{1}{|G|}$$

$\cancel{\#} \langle e_{S^2}, e_{S^2} \rangle = |G|$ since $\lambda = 1$

$$= Z(B \setminus B^3)(e_{S^2}) \cdot \frac{1}{|G|} \quad \text{since } Z(B^3)(e_{S^2}) = 1.$$

$\partial Y \neq \emptyset$

$Y \times I \cong$ 1-handle-body



$A(Y; c)$

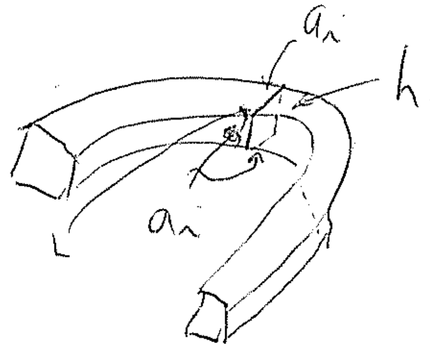
$a_i \in G \quad \langle (a_i), (b_i) \rangle = \begin{cases} 0 & \text{unless } a_i = b_i \ \forall i \\ 1, & a_i = b_i \ \forall i \end{cases}$

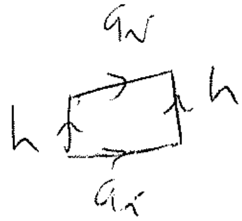
$$\langle Q_1, Q_2 \rangle = \frac{1}{|Aut(Q)|}$$

Y closed connected

$(Y \setminus D^2) \times I \qquad D^2 \times I \xrightarrow{gl} Y \times I$

$A(Y; c)$ with $c =$ const. map





if $[h, a_i] \neq 1$ get 0.

$$\sum_{h \in G} \rightsquigarrow \sum_{\substack{h \text{ s.t.} \\ 1 = [h, a_i] \ \forall i}}$$

$Aut(Q)$.

So inner product is $|Aut(Q)|$

$$\langle \delta_{Q_1}, \delta_{Q_2} \rangle = \frac{1}{|Aut(Q)|}$$