Ricci Flow and Renormalization II

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Setup:

M = Riemannian manifold

$\{(G_{ij}, \varphi) : \text{Riemannian metric on } M,\ \varphi : M \to \mathbb{R}\}$

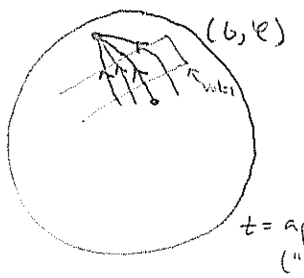
Sigma Models: 2D quantum field theory

$X : \Sigma \to M$ ← worldsheet (Σ, g_{ab})

action $I(X) = \frac{1}{4\pi\alpha'}\left(\|dX\|^2_{L^2} + \alpha' \int \varphi \circ X \, R \, d\mathrm{vol}_\Sigma\right)$, $R = R(g_{ab})$

$$= \frac{1}{4\pi\alpha'} \int d^2x \sqrt{g}\left[g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X)\right] + \alpha' R \varphi(X)$$

$$\int_{\mathrm{Maps}} F(X) e^{-I(X)}$$



$t \mapsto t + \Delta t$

$G_{ij} \mapsto G_{ij}(t + \Delta t)$

$\varphi(t) \mapsto \varphi(t + \Delta t)$

$\frac{dG_{ij}}{dt} = \square$

$\frac{d\varphi}{dt} = \square$

t = approximation scale ("RG time")

$$\frac{dG_{\mu\nu}}{dt} = -\beta^G_{\mu\nu}$$

$$\frac{d\varphi}{dt} = -\beta^\varphi$$

Result: $\beta^G_{\mu\nu} = \alpha' R_{\mu\nu} + \frac{1}{2}(\alpha')^2 R_{\mu\lambda\rho\sigma} R_\nu{}^{\lambda\rho\sigma} + O((\alpha')^3 R^3)$

$\beta^\varphi = C_0 - \frac{1}{2}\alpha' D^2 \varphi + \frac{1}{16}\alpha^2 R_{\lambda\mu\nu\rho} R^{\lambda\mu\nu\rho} + O((\alpha')^3 R^3)$

← bosonic: $\frac{1}{6}(D-26)$

$\overline{\beta^G_{\mu\nu}} = \beta^G_{\mu\nu} + \nabla_\mu M_\nu + \nabla_\nu M_\mu$

$\overline{\beta^\varphi} = \beta^G + M^\mu \partial_\mu \varphi$

where $M^\mu = \alpha' \partial_\mu \varphi + W_\mu(G)$ and $W_\mu = \frac{1}{2}(\alpha')^3 \partial_\mu (R_{\lambda\nu\gamma\rho} R^{\lambda\nu\gamma\rho}) + \dots$

$S(G, \varphi) : \{(G, \varphi)\} \to \mathbb{R}$

$$\frac{\delta S}{\delta G_{\mu\nu}} = K_{\mu\nu}{}^{\lambda\pi}\, \overline{\beta}^G_{\lambda\pi} + \dots$$

$$\frac{\delta S}{\delta \varphi} = K^{\mu\nu}\, \overline{\beta}^G_{\mu\nu} + \dots$$

"Central charge action": $S = \int e^{-2\varphi}\left[C_0 - \alpha'\left(\underline{\tfrac{1}{4}R + \partial_\mu\varphi\, \partial^\mu\varphi}\right) + O(\alpha'^2)\right] d\mathrm{vol}_M$

" $\int d^D y \sqrt{G}\, G^{\mu\nu} \dots$

$\varphi_i \equiv (G_{\mu\nu}, \varphi)$

$\frac{\delta S}{\delta \varphi_i} = K_{ij} \overline{\beta}^j \rightsquigarrow \overline{\beta}^i = K^{ij} \frac{\delta S}{\delta \varphi_j}$

Problem: $-K_{ij}$ is not positive definite

$$\frac{\delta S}{\delta \varphi} = -2\sqrt{G}\, e^{-2\varphi}\, \underbrace{\bar{\beta}^{\varphi}}_{= C_0 - \alpha'(\ \ ) \dots}$$

Perelman's Idea: $S(G,\varphi)$ should be extremized relative to volume $=1$

$$\boxed{\int d^D x \sqrt{G}\, e^{-2\varphi} = 1}$$

$$\hat{S}(G,\varphi) = S(G,\varphi) + \lambda\left(\int d^D x \sqrt{G}\, e^{-2\varphi} - 1\right)$$

$$= \int d^D x \sqrt{G}\, e^{-2\varphi} (\bar{\beta}^{\varphi}) + \lambda\left(\int \dots\right)$$

$$\hat{S} = S(C_0 \to C_0 + \lambda) - \lambda$$

$$\frac{\delta \hat{S}}{\delta \varphi} = 0 \Rightarrow \bar{\beta}^{\varphi} + \lambda = 0 \Rightarrow \lambda = -\bar{\beta}^{\varphi}$$

$$\frac{\delta \hat{S}}{\partial \varphi^i} = K_{ij}\, \bar{\beta}^j \quad \Rightarrow -K_{ij} \text{ positive definite}$$