Complex Manifold With Torsion

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Kähler geometry:

Basic foundation of Kähler geometry is that the holomorphic group of the metric is $U(n)$ and the Laplacian operator acting on differential form Ω^m commutes with the projection operator: $\Omega^m \to \Omega^{p,q}$ with $p + q = m$ based on the fact that the Kähler form $\sqrt{-1}\sum g_{i\bar{j}} dz_i \wedge d\bar{z}_j$ is covariant constant.

Calabi-Yau manifolds are a special class of Kähler manifolds M^n with Ricci curvature $=0$.

If M is simply connected, there is a nowhere vanishing holomorphic n-form Ω. This n-form Ω is also covariant constant.

In this case the canonical line bundle of M is trivial and the Dirac spinors can be identified with differential forms and the Dirac operators are $\bar{\partial}$ operator.

ω, Ω are parallel spinors. In the string theory, these two parallel spinors provide $N = 2$ supersymmetries. It creates possibility of Mirror symmetries and many important consequences in algebraic geometry of Calabi-Yau manifolds. A most notable contribution was the (rigorous) proof of Givental, Lian-Liu-Yau for the formula of Candelas et al on counting number of rational curves of various degrees.

It was conjectured that for each dimension, there are only a finite number of deformation types of CY manifolds. The most interesting CY manifolds have dimension three. There are more than 10 thousand of such deformation types. It would be important to give a unified understanding of such manifolds. They may all be complete intersections in some toric manifolds.

M. Reid made a proposal based on the construction of Clemens-Friedman. Clemens wanted to take a rational curve in CY manifold M whose normal bundle is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and contract such a curve to a rational double point. Friedman proposed the condition to deform such a manifold to be a smooth complex manifold. By blowing down enough such rational curves, $H^2(M)$ can be killed and we end up with a complex manifold which is not Kähler and is diffeomorphic to a connected sum of $S^3 \times S^3$.

Reid conjectured that one can connect any CY threefold to another one through such conifold transitions. It is a nice picture and can be checked in many cases. However, one needs to understand the geometry of such non-Käker manifolds.

In order to do this, we find that the most suitable structure is Hermitian metric with torsion introduced by Strominger.

The important point here is that supersymmetry still exists. (There are parallel spinors.) There are four equations in Strominger system. (We shall write them explicitly.)

The last equation of Strominger system is existence of a certain Hermitian form ω so that $d(\omega^2) = 0$. Such a class of metrics were studied by M.L.Michelsohn and L.Alessandrini-G.Bassanelli. They called them balanced metrics.

They proved that existence of balanced metric respects fiber bundle construction quite well and also it respects birational transformation.

Hence we believe that the balanced manifold and the Strominger system is a good class of manifolds. We shall now discuss how such equations arose in string theory and how we proved existence.

In the original proposal for compactification of superstring, Candelas, Horowitz, Strominger and Witten took the metric product of a maximal symmetric four dimensional space-time M with a six dimensional Calabi-Yau vacuum X as the ten dimensional space-time; they identified the Yang-Mills connection with the $SU(3)$ connection of the Calabi-Yau metric and set the dilaton to be a constant. Adapting the second author's suggestion of using Uhlenbeck-Yau's theorem on constructing Hermitian-Yang-Mills connections over stable bundles, Witten and later Horava-Witten proposed to use higher rank bundles for strong coupled heterotic string theory so that the gauge groups can be $SU(4)$ or $SU(5)$.

A. Strominger

Compactification Ansatze

$$g^{0}_{MN}(x,y) = e^{2D(y)} \begin{pmatrix} \widehat{g}^{(y)}_{mn} & 0 \\ 0 & \widehat{g}^{(x)}_{\mu\nu} \end{pmatrix}$$

where y is coordinate on internal space K and x is coordinate on some maximally symmetric spacetime.

Spacetime supersymmetry forces:

D to be dilation field.

A supersymmetric configuration requires existence of Mogorava–Weyl spinor ε^0 so that

$$\delta\psi^\circ_M = \nabla^\circ_M \varepsilon^\circ + \frac{1}{48}(\Upsilon^\circ_M H^\circ - 12 H^\circ_M)\varepsilon^\circ = 0$$

$$\delta\lambda^\circ = (\not\nabla^\circ \phi)\varepsilon^\circ + \frac{1}{24} e^{2\phi} H^\circ \varepsilon^\circ = 0$$

$$\delta\chi^\circ = e^\phi F_{MN} \Gamma^{\circ MN} \varepsilon^\circ$$

where

ψ° is the gravitano

λ° is the dilatino

χ° is the gluino.

Write

$$
\begin{aligned}
g_{MN} &= e^{-2\phi} g^{\circ}_{MN} \\
\varepsilon &= e^{-\phi/2} \varepsilon^{\circ} \\
\psi_M &= e^{-\phi/2} \left(\psi^{\circ}_M - \frac{1}{2} \Upsilon^{\circ}_M \lambda^{\circ} \right) \\
\lambda &= e^{\phi/2} \lambda^{\circ} \\
\Upsilon_M &= e^{\phi} \Upsilon^{\circ}_M .
\end{aligned}
$$

Then

$$
\begin{aligned}
\nabla_M \varepsilon - \frac{1}{4} H_M \varepsilon &= 0 \\
(\nabla\!\!\!/ \phi)\varepsilon + \frac{1}{24} H \varepsilon &= 0.
\end{aligned}
$$

Then there exist positive and negative chirality spinors $\eta_\pm$ that are H-covariantly constant. (The three form H_{MNP} defines a connection. Note that we assume ϕ depends only on K and the components of H tangent to the maximally symmetric spacetime vanish.)

We normalize

$$\eta_\pm^\dagger \eta_\pm = 1.$$

Then

$$J_m^n = \sqrt{1}\,\eta_+^\dagger \Gamma_m^n \eta_+$$

is an almost complex structure.

J_m^n is H-covariant constant

$$\nabla_m J_n^p - H^p_{sm} J^s_n - H^s_{mn} J^p_s = 0.$$

It is integrable.

The Kähler form is

$$\begin{aligned} \omega &= \frac{1}{2} J_m^n g_{np}\, dx^m \wedge dx^p \\ &= \sqrt{-1}\, g_{a\bar{b}}\, dz^a \wedge d\bar{z}^b \\ H &= -\frac{\sqrt{-1}}{2}(\bar{\partial} - \partial)\omega. \end{aligned}$$

The holomorphic n form is given by

$$\Omega = e^{8\phi}\eta_-^\dagger \Gamma_{a_1 a_2 a_3}\eta_-\, dz^{a_1}\cdots dz^{a_n}.$$

It turns out that

$$\phi - \frac{1}{8}\ln\|\Omega\| \quad \text{is a constant.}$$

Since $\delta\lambda = 0$,

$$-8\nabla_m\phi + J_m^n \nabla_p J_n^p = 0$$

and

$$d^*\omega - \sqrt{-1}\,(\bar{\partial} - \partial)\ln\|\Omega\|_\omega = 0$$

and we arrive at the equation of Strominger.

(1) $F_h \wedge \omega^2 = 0$

(2) $F_h^{2,0} = F_h^{0,2} = 0$

(3) $\partial\bar{\partial}\omega = \sqrt{-1}Tr(F_h \wedge F_h) - \sqrt{-1}Tr R_g \wedge R_g$

(4) $d^*\omega = \sqrt{-1}(\bar{\partial} - \partial)\log\|\Omega\|_\omega$

I. Li-Yau: Perturbation method

Let E_s be smooth family of holomorphic vector bundles over a Calabi-Yau space X. Let h_0 be a Hermitian–Yang–Mills connection on E_0.

Then we like to extend h_0 to be a smooth family of Hermitian–Yang–Mills connection.

The interesting case is when h_0 is reducible.

Let (X, ω_0) be Kähler.

Let (E_1, D_1'') and (E_2, D_2'') be degree zero and slope-stable vector bundles.

Let h_1 and h_2 be the Hermitian metrics on E_1 and E_2 respectively.

The $h_1 \oplus th_2$ is still a Hermitian metric corresponds to the connection $D_0'' = D_1'' \oplus D_2''$.

Suppose we are given a deformation of holomorphic structure D''_s of D''_0. Then Kodaira–Spencer identifies the first order deformation of D''_s at 0 to an element

$$k \in H^1(X, \varepsilon^* \otimes \varepsilon)$$

where ε is the sheaf of holomorphic section s of (E, D''_0).

Therefore

$$k \in \oplus_{i,j=1}^2 H^1(\varepsilon_i^* \otimes \varepsilon_j).$$

Theorem. Suppose k_{12} and k_{21} are nonzero. Then there is a unique t so that for s sufficiently small $h_0(t) = h_1 \oplus e^t h_2$ extends to a smooth family of Hermitian–Yang–Mills metric on (E, D''_s).

The fourth equation of Strominger system is equivalent to

$$d\left(\|\Omega\|_\omega \omega^2\right) = 0.$$

Let $\mathcal{H}(X)$ be the cone of positive definite Hermitian form on X.

Let $\mathcal{H}(E)_0$ be the space of determinant one Hermitian metric on the bundle E (i.e., the induced metric on $\wedge^r E \simeq \mathbb{C}_X$ is the constant one metric).

We define

$$L = L_1 \oplus L_2 \oplus L_3 :$$
$$\mathcal{H}(E)_0 \times \mathcal{H}(X) \longrightarrow$$
$$\Omega^{3,3}(\mathrm{End}^0 E) \oplus \mathrm{Im}\,\sqrt{-1}\partial\bar{\partial} \oplus \mathrm{Im}\, d_0^*$$

where

$$L_1(H,\omega) = \sqrt{-1}F_H \wedge \omega^2$$
$$L_2(H,\omega) = \sqrt{-1}\partial\bar{\partial}\omega + \mathrm{tr}_E(F_H \wedge F_H)$$
$$- \mathrm{tr}_T(R_g \wedge R_g)$$
$$L_3(H,\omega) = *_0 d\left(\|\Omega\|_\omega \omega^2\right).$$

We shall apply implicit function theorem to L.

Fix a determinant one Hermitian metric $\langle\, , \rangle$ on E we can write other determinant one Hermitian metric on E by a unique positive definite $\langle\, , \rangle$ Hermitian symmetric endomorphism H of Z satisfying $\det H = 1$.

Such spaces H will be denoted by $\Gamma(\mathrm{End}_h^+ E)$, identity $I \in \Gamma(\mathrm{End}_h^+ E)$.

The tangent space at I is $\Gamma(\mathrm{End}_h^0 E)$ traceless symmetric endomorphisms of E.

$$\delta L_1(I,\omega_0)(\delta h,\delta\omega)$$

$$= D''D'_H\delta h + 2F_H \wedge \omega_0 \wedge \delta\omega$$

$$\delta L_2(I,\omega_0)(\delta h,\delta\omega)$$

$$= \sqrt{-1}\partial\bar{\partial}(\delta\omega) + 2(\mathrm{tr}_E\delta F_I(\delta h) \wedge F_I)$$

$$- \mathrm{tr}_T\delta R_{g_0}(\delta g) \wedge R_{g_0}$$

$$\delta L_3(I,\omega_0)(\delta h,\delta\omega)$$

$$= 2d_0^*(\delta\omega) - d_0^*((\delta\omega,\omega_0)\omega_0).$$

Construction of irreducible solution to Strominger's system perturbatively.

Start with a Calabi–Yau manifold,

$$(E, D_0'') = \mathbb{C}_X^{\oplus(r-3)} \oplus T_X,$$

the metric is identified with $I : E \longrightarrow E$.

For all $c > 0$, $(I, c\omega_0)$ is a solution to $L = 0$.

Let

$$W_1 = \Omega^{3,3}_{\mathbb{R}}(\mathrm{End}^0_h E)_{L^p_{k-2}}$$

$$W_2 = (\mathrm{Im}\,\sqrt{-1}\,\partial\bar\partial)_{L^p_{k-2}} \oplus (\mathrm{Im}\,d_0^*)_{L^p_{k-1}}$$

$$V_0 = \Big\{ A \oplus aI_{T_X} \mid \quad A \in \mathrm{End}\,\mathbb{C}_X^{\oplus(r-3)} \text{ are constant matrices such that } A = A^{-t},\ \mathrm{tr}\,A + 3a = 0 \Big\}$$

$$V_1 = \omega_0^3 \otimes V_0.$$

Then $\exists\, C > 0$ such that for all $c > C$,

$$\delta L_1(I, c\omega_0) \oplus \delta L_2(I, c\omega_0) \oplus \delta L_3(I, c\omega_0)$$

$$: \Gamma(\mathrm{End}^0_h E)_{L^p_k} \oplus \Omega^{1,1}(X) \longrightarrow W_1/V_1 \oplus W_2$$

is surjective.

Theorem. Let X be a Calabi–Yau 3-fold with ω a Ricci flat Kähler form. Let D''_s be a smooth deformation of holomorphic structure D''_0 on $E = \mathbb{C}_X \oplus T_X$. Suppose the associated cohomology classes $[C_{12}]$ and $[C_{21}]$ are non-zero. Then for sufficiently large c there is a family of pairs of Hamiltonian metrics and Hamiltonian forms (H_s, ω_s) for $0 \leq s < \varepsilon$ so that

1. $\omega_0 = c\omega$ and the harmonic part of ω_s is equal to $c\omega$.

2. The pair (H_s, ω_s) is a solution to Strominger's system for the holomorphic vector bundle (E, D''_s).

Let

$$D''_s = D''_0 + A_3, \quad A_s \in \Omega^{0,1}(\mathrm{End}\, E)$$

$$A_0 = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \in \Omega^{0,1}(\mathrm{End}\, E).$$

We can assume C_{ij} are D''_0 harmonic. Since $H^1(X, \mathcal{O}_X) = 0$, $C_{11} = 0$.

In general, we consider the $r+3$ holomorphic vector bundle $\mathbb{C}_X^{\oplus r} \oplus T_X$. We also have

$$D_0'' = \begin{pmatrix} 0 & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

where

$$\begin{aligned} C_{12} =& (\alpha_1, \ldots, \alpha_r)^t \in \Omega^{0,1}(T_X)^{\oplus j} \\ C_{21} =& (\beta_1, \ldots, \beta_r) \in \Omega^{0,1}(T_X^r)^{\oplus j} \\ C_{22} \in& \Omega^{0,1}(\text{End}\, T_X). \end{aligned}$$

Suppose $[\alpha_1], \ldots, [\alpha_r] \in H^1(X, T_X^*)$ are linearly independent and $[\beta_1], \ldots, [\beta_r] \in H^1(X, T_X^*)$ are linearly independent. Then the above theorem holds.

Example

Consider

$$X = \{z_0^5 + \cdots + z_x^5 = 0\} \in \mathbb{P}^4$$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & T_X & \longrightarrow & T_X\mathbb{P}^4 & \longrightarrow & \mathcal{O}_X(5) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & F & \longrightarrow & \mathcal{O}_X(1)^{\oplus 5} & \longrightarrow & \mathcal{O}_X(5) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathcal{O}_X & = & \mathcal{O}_X & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Here F is the cokernel of $\mathcal{O}_X(1)^{\oplus 5} \longrightarrow \mathcal{O}_X(5)$ and fill in

$$0 \longrightarrow \mathcal{O}_X \longrightarrow F \longrightarrow TX \longrightarrow 0.$$

The above sequence is a non-split extension.

Making use of this element in $\text{Ext}^1(T_X, \mathcal{O}_X)$ we can form a deformation of holomorphic structure D_t'' of that $C_{12} \neq 0$, $C_{21} \neq 0$.

Hence we have proved:

Let X be a smooth quintic threefold and ω be any Kähler form on X. Then for large $c > 0$, there is a smooth deformation of $\mathbb{C}_X \oplus T_X$ so that for small s, there are pairs (H_s, ω_s) of Hamiltonian metrics on E and Hamiltonian forms ω_s on X. That solves Strominger's system.

For the Calabi–Yau manifold with three generations that I constructed:

$$X \subset \mathbb{P}^3 \times \mathbb{P}^3$$

given by

$$\sum x_i^3 = 0$$
$$\sum y_i^3 = 0$$
$$\sum x_i y_i = 0$$

quotient by $\mathbb{Z}_3$. One can also construct irreducible solution to Strominger's system on $TX \oplus \mathbb{C}_X^{\oplus 2}$.

II. Fu-Yau: Non-Kähler manifolds

K. Becker, M. Becker, K. Dasgupta and P. S. Green, *Compactifications of heterotic strings on non-Kähler complex manifolds, I.* JHEP **0304** (2003), 007, hep-th/0301161

E. Goldstein and S. Prokushkin, *Geometric model for complex non-Kähler manifolds with* $SU(3)$ *structure*, hep-th/0212307.

Let (S, ω_S, Ω_S) be the K3 surface. Let $\frac{\omega_1}{2\pi}, \frac{\omega_2}{2\pi} \in H^2(S, \mathbb{Z})$ and let ω_1 and ω_2 be anti-self-dual (1,1)-forms. Then there is a non-Kähler manifold X such that $\pi : X \to S$ is a holomorphic T^2 fibration over S.

If we write locally $\omega_1 = d\alpha_1$ and $\omega_2 = d\alpha_2$, then there are coordinates of the fibers T^2 x and y such that $dx + \pi^*\alpha_1$ and $dy + \pi^*\alpha_2$ are globally defined 1-forms on X.

Let

$$\theta = dx + \pi^*\alpha_1 + \sqrt{-1}(dy + \pi^*\alpha_2).$$

Then hermitian form on X is

$$\omega_0 = \pi^*\omega_S + \frac{\sqrt{-1}}{2}\theta \wedge \bar{\theta}$$

and holomorphic 3-form is

$$\Omega = \pi^*\Omega_S \wedge \theta.$$

ω_0 satisfies the forth equation $d(\| \Omega \|_{\omega_0} \omega_0^2) = 0$.

Let u be any smooth function on S and let

$$\omega_u = \pi^*(e^u\omega_S) + \frac{\sqrt{-1}}{2}\theta \wedge \bar{\theta}.$$

Then ω_u is a Hermitian metric on X and (ω_u, Ω) also satisfies $d(\| \Omega \|_{\omega_u} \omega_u^2) = 0$.

As ω_1 and ω_2 are harmonic, $\bar{\partial}\omega_1 = \bar{\partial}\omega_2 = 0$. According to $\bar{\partial}$-Poincare Lemma, we can write ω_1 and ω_2 locally as

$$\omega_1 = \bar{\partial}\xi = \bar{\partial}(\xi_1 dz_1 + \xi_2 dz_2)$$

and

$$\omega_2 = \bar{\partial}\zeta = \bar{\partial}(\zeta_1 dz_1 + \zeta_2 dz_2),$$

where (z_1, z_2) is the local coordinate on S. Let

$$B = \begin{pmatrix} \xi_1 + \sqrt{-1}\zeta_1 \\ \xi_2 + \sqrt{-1}\zeta_2 \end{pmatrix}.$$

If we let R_u be the curvature of Hermitian connection of Hermitian metric ω_u on the holomorphic tangent bundle, then

$$\begin{aligned}\mathrm{tr} R_u \wedge R_u =& \mathrm{tr} R_S \wedge R_S + 2\partial\bar{\partial}u \wedge \partial\bar{\partial}u \\ &+ 2\partial\bar{\partial}(e^{-u}\mathrm{tr}(\bar{\partial}B \wedge \partial B^* \cdot g^{-1})).\end{aligned}$$

So the third equation in Strominger's system can be reduced to

$$\begin{aligned} & \sqrt{-1}\partial\bar{\partial}e^u \wedge \omega_S - 2\partial\bar{\partial}u \wedge \partial\bar{\partial}u \\ & - 2\partial\bar{\partial}(e^{-u}\mathrm{tr}(\bar{\partial}B \wedge \partial B^* \cdot g^{-1})) \\ = \quad & \mathrm{tr} R_S \wedge R_S - \mathrm{tr} F_H \wedge F_H \\ & - (| \omega_1 |^2 + | \omega_2 |^2)\frac{\omega_S^2}{2!}. \end{aligned} \tag{1}$$

Since $\mathrm{tr} F_H \wedge F_H \geq 0$ and $\mathrm{tr} R_S \wedge R_S = 0$ in the case of T^4, we obtain

Proposition. *There is no solution of Strominger's system on the torus bundle X over T^4 if we use the ansatz:*

$$e^u \omega_S + \frac{\sqrt{-1}}{2} \theta \wedge \bar{\theta}.$$

We consider the case of $K3$ surface. Let (E, H) be the Hermitian-Yang-Mills vector bundle over S with the gauge group $SU(r)$. Then $(V = \pi^*E, H)$ is also the Hermitian-Yang-Mills vector bundle over X. We can consider the equation (1) as the equation on the $K3$ surface S. Integrating equation (1) over S,

$$\int_S \{\mathrm{tr} R_S \wedge R_S - \mathrm{tr} F_H \wedge F_H\}$$
$$= \int_S (\| \omega_1 \|_{\omega_S}^2 + \| \omega_2 \|_{\omega_S}^2) \frac{\omega_S^2}{2!}.$$

We use $Q(\frac{\omega_i}{2\pi})$ to denote the intersection number of anti-self-dual (1,1)-form $\frac{\omega_i}{2\pi}$. As $\int_S \mathrm{tr} R_S \wedge R_S = 24$, the above condition can be written as

$$2(24 - c_2(E)) = -\left(Q\left(\frac{\omega_1}{2\pi}\right) + Q\left(\frac{\omega_2}{2\pi}\right)\right). \tag{2}$$

Certainly We can choose ω_1 and ω_2 and $SU(r)$ vector bundle E such that they satisfy the condition (2). Then there is a smooth function μ such that

$$\begin{aligned} &\mathrm{tr} R_S \wedge R_S - \mathrm{tr} F_H \wedge F_H \\ &- \left(\|\omega_1\|^2 + \|\omega_2\|^2\right) \frac{\omega_S^2}{2!} = -\mu \frac{\omega_S^2}{2!}. \end{aligned}$$

So we obtain the following equation:

$$\sqrt{-1}\partial\bar{\partial}e^u \wedge \omega_S - \partial\bar{\partial}u \wedge \partial\bar{\partial}u$$
$$-\partial\bar{\partial}(e^{-u}\mathrm{tr}(\bar{\partial}B \wedge \partial B^* \cdot g^{-1})) + \mu\frac{\omega_S^2}{2!} = 0. \tag{3}$$

In particular, when $\omega_2 = n\omega_1$, $n \in \mathbb{Z}$, we have

$$\mathrm{tr}(\bar{\partial}B \wedge \partial B^* \cdot g^{-1}) = \sqrt{-1}\frac{1+n^2}{4} \parallel \omega_1 \parallel_{\omega_S}^2 \omega_S.$$

Hence in this case, if we set $f = \frac{1+n^2}{4} \parallel \omega_1 \parallel_{\omega_S}^2$, we can rewrite equation (3) as the standard complex Monge-Ampere equation:

$$\Delta(e^{-u} - fe^{-u}) + 8\frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}} + \mu = 0.$$

We solve the equation (3) by the continuity method and get

Theorem. The equation (3) has a smooth solution u such that

$$\begin{aligned}\omega' = & e^u \omega_S + 2\sqrt{-1}\partial\bar{\partial}u \\ & - \sqrt{-1}e^{-u}\mathrm{tr}(\bar{\partial}B \wedge \partial B^* \cdot g^{-1})\end{aligned}$$

is a Hermitian metric on S.

Theorem. Let S be a $K3$ surface with Calabi-Yau metric ω_S. Let ω_1 and ω_2 be anti-self-dual $(1,1)$-forms on S such that $\frac{\omega_1}{2\pi} \in H^2(S,\mathbb{Z})$ and $\frac{\omega_2}{2\pi} \in H^2(S,\mathbb{Z})$. Let X be a T^2-bundle over S constructed by ω_1 and ω_2. Let E be a stable bundle over S with the gauge group $SU(r)$. Suppose ω_1, ω_2 and $c_2(E)$ satisfy the topological constraint (2). Then there exist a smooth function u and a Hermitian-Yang-Mills metric H on E such that $(V = \pi^*E, \pi^*F_H, X, \omega_u)$ is a solution of Strominger's system.

For simplicity, we consider the equation

$$\Delta(e^{-u} - fe^{-u}) + 8\frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}} + \mu = 0.$$

We impose the following elliptic condition

$$\omega' = (e^{u} + fe^{-u})\omega + 2\sqrt{-1}\partial\bar{\partial}u > 0$$

and the normalization condition

$$\left(\int_S \exp(-4u)\frac{\omega_S^2}{2!}\right)^{\frac{1}{4}} = A, \qquad \int_S 1\frac{\omega_S^2}{2!} = 1.$$

Zeroth order estimate

Let $P = 2g'^{i\bar{j}}\frac{\partial^2}{\partial z_i \partial \bar{z}_j}$. We have two methods to calculate

$$\int_S P(e^{ku})\frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}}\frac{\omega_S^2}{2!}.$$

Then using the Sobolev inequality, Moser iteration and Poincare inequality, we can get

Proposition. If $A < 1$, then there is a constant C_1 which depends on f, μ and the Sobolev constant of ω_S such that

$$\inf_S u \geq -\ln(C_1 A).$$

Moreover, if A is small enough such that $A < (C_1)^{-1}$, then there is an upper bound of $\sup_S u$ which depends on f, μ, Sobolev constant of ω_S and A.

An estimate of the determinant

We should estimate the lower bound of determinant

$$F = \frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}}.$$

We apply maximum principle to function

$$G = 1 - e^{-u} \mid \nabla u \mid^2 + 2e^{-\epsilon u} - 2e^{-\epsilon \inf u}$$

and get

Proposition. Given any constant $\kappa \in (0,1)$, we fix some positive constant ϵ satisfying

$$\epsilon < \min\{1, 2^{-1}\kappa\}.$$

Suppose that A satisfies

$$A < \min\{1, C_1^{-1}, \{2(1+\sup f)\}^{-\frac{1}{2}} C_1^{-1}, \left(\frac{1-\kappa}{2C_3}\right)^{\frac{1}{\epsilon}} C_1^{-1}, \frac{3-6\epsilon}{C_4} C_1^{-1}, C_5\},$$

where C_3 and C_4 depend on f, μ, and C_4 also depends the curvature bound of ω_S; C_5 depends on κ, ϵ and C_3. Then $F > \kappa e^{2u} \geq \kappa (C_1 A)^{-2}$.

Second order estimate

Since

$$e^u + fe^{-u} + \triangle u \geq F^{\frac{1}{2}} > \kappa^{\frac{1}{2}}(C_1 A)^{-1} > 0,$$

it is sufficient to have an upper estimate of $e^u + fe^{-u} + \triangle u$. Applying maximum principle to function

$$e^{-\lambda_1 u + \lambda_2 |\nabla u|^2} \cdot (e^u + fe^{-u} + \triangle u),$$

where λ_1 and λ_2 are positive constants and will be determined, we can get the estimate of second order estimate.

Third order estimate

Let

$$
\begin{aligned}
\Gamma &= g^{i\bar{j}} g^{k\bar{l}} u_{,ik} u_{,\bar{j}\bar{l}} \\
\Theta &= g'^{i\bar{r}} g'^{s\bar{j}} g'^{k\bar{t}} u_{,i\bar{j}k} u_{,\bar{r}s\bar{t}} \\
\Xi &= g'^{i\bar{j}} g'^{k\bar{l}} g'^{p\bar{q}} u_{,ikp} u_{,\bar{j}\bar{l}\bar{q}} \\
\Phi &= g'^{i\bar{j}} g'^{k\bar{l}} g'^{p\bar{q}} g'^{r\bar{s}} u_{,i\bar{l}pr} u_{,\bar{j}k\bar{q}\bar{s}} \\
\Psi &= g'^{i\bar{j}} g'^{k\bar{l}} g'^{p\bar{q}} g'^{r\bar{s}} u_{,i\bar{l}p\bar{s}} u_{,\bar{j}k\bar{q}r},
\end{aligned}
$$

where indices preceded by a comma indicate covariant differentiation with respective to the metric ω_S.

We apply maximum principle to the function

$$(\kappa_1+\triangle u)\Theta+\kappa_2(m+\triangle u)\Gamma+\kappa_3 \mid \bigtriangledown u \mid^2 \Gamma+\kappa_4\Gamma,$$

where all κ_i are positive constants and will be determine; and m is a fixed constant such that $m+\triangle u>0$. Then we can get the third order estimate.

Topology of the total space:

$$\begin{aligned}
(1)\ h^{0,1}(X) &= h^{0,1}(S)+1 \\
(2)\ h^{1,0}(X) &= h^{1,0}(S) \\
(3)\ b_1(X) &= b_1(S)+1 \\
(4a)\ b_2(X) &= b_2(S)-1, \\
& \quad \text{if } \omega_1 \text{ is a multiple of } \omega_2; \\
(4b)\ b_2(X) &= b_2(S)-2, \\
& \quad \text{if } \omega_1 \text{ is not a multiple of } \omega_2.
\end{aligned}$$

Becker-Becker–Tseng: Duality argument

The supersymmetric heterotic compactification with non-zero H_3 flux with metric

$$ds^2 = e^{2\phi} ds^2_{M_4} + (dx + \alpha_1)^2 + (dy + \alpha_2)^2$$

has a $M-$theory dual solution for base $M_4 = K3$ and not for $M_4 = T^4$. Since the supergravity duality mapping does not affect the base, the lack of a $M-$theory dual strongly suggested that there is no $H_3 \neq 0$ heterotic solution with base $M_4 = T^4$.

For $M_4 = K3$, the heterotic solution is dual to M-theory on $K3 \times K3$, with the second $K3$ taken as a T^4/Z_2 orbifold. (To be precise, the metric is conformal $K3 \times K3$.) Starting from $M-$theory on $K = K3 \times K3$ with non-zero flux, the series of dualities ending in the heterotic solution are roughly as follows. Treat the second $K_3 = T^4/Z_2$ as an elliptic fibration over P_1. Reducing the T^2 fiber to zero size, we obtain type IIB theory on $K3 \times T^2/Z_2$ where here $Z_2 = \Omega(-1)^{F_L} I_{89}$ with $I_{89} : (x, y) \to (-x, -y)$ and Ω being the world-sheet parity operator.

Applying further two $T-$dualities, one in each direction of T^2/Z_2, results in the dual type I theory on $K3$ times a T^2 fibration. Finally, an S-duality takes the type I background to the above heterotic solution. Notice that duality at each step leaves the base $K3$ unchanged. Therefore, if we replace the base with T^4, the dual M-theory solution should be $M-$theory on $T^4 \times K3$ with non-zero flux. However, such an $M-$theory solution is inconsistent as a supersymmetric background.

To see this, we study the constraints on M-theory backgrounds imposed by supersymmetry and the equation of motion. The 11 dimensional M-theory has 4-form denoted as G_4. One of the equation of motion has the form,

$$d * G_4 = -\frac{1}{2} G_4 \wedge G_4 - (2\pi)^2 \beta X_8$$

where $X_8 = -\frac{1}{768}(\text{tr}R^2)^2 + \frac{1}{192}\text{tr}R^4$ and β which contains $'$ is often conventionally set to one. The above equation can be integrated over the compact four-fold K to give the condition

$$\frac{1}{8\pi} \int_K G_4 \wedge G_4 = \frac{\chi(K)}{24}$$

where χ is the Euler number. (If additional N $M2$-brane sources are present, we must add $a+N$ term on the LHS of the above equation.

Now, supersymmetry requires that G_4 is $(2,2)$ and primitive and thus $*G_4 = G_4$. Therefore, the integral $\int G_4 \wedge G_4$ is positive definite and is zero only if $G_4 = 0$. We therefore see that a nonzero G_4 is allowed for $K = K3 \times K3$. However, for $K = T^4 \times K3$, $\chi(K) = 0$ and therefore G_4 must vanish. But under the duality mapping, the heterotic H_3 (and also F_2) is associated with the $M-$theory G_4. In particular, a zero G_4 after duality leads to a zero H_3 in the heterotic theory. Thus, from the duality analysis, we do not expect a heterotic solution with base $M_4 = T^4$ and $H_3 \neq 0$.