

Regularization of σ -models

Standard example " φ^3 theory" scalar field in D -dims

$$\frac{1}{2} S = \int \left(\frac{1}{2} (\nabla\varphi)^2 - \frac{m^2}{2} \varphi^2 + \frac{g}{3!} \varphi^3 \right) d^D x$$

idea $\int \mathcal{D}\varphi \mathcal{O}_1(\varphi) \dots \mathcal{O}_N(\varphi) e^{-S(\varphi)}$ path integral

expand as a function of g
asymptotic expansion of integral (Feynman)

- Fourier form to momentum variables
- Σ (terms labeled by Feynman diagrams)

typical term $\int \frac{d^D q}{(q^2 + m^2)(q-k^2 + m^2)}$

3 Regularization methods

1) Replace \mathbb{R}^D with lattice \mathbb{Z}^D

→ Discrete analogues of quantities in theory

scale lattice spacing $a \rightarrow 0$

2) Momentum cutoff $\int_{\mu}^{\Lambda} \frac{d^D q}{(q^2 + m^2) \dots}$ $\Lambda \rightarrow \infty$ divergent
 μ sets scale

3) Dimensional regularization - applies term by term in Feynman expansion

$$\int_{\mu}^{\Lambda} \frac{d^D q}{|q|^2 + m^2} = \int_0^{\infty} d\sigma \int_{\mu \rightarrow 0}^{\Lambda \rightarrow \infty} e^{-\sigma(|q|^2 + m^2)} d^D q$$

$$= \int_0^{\infty} 2^{-D} \pi^{-D/2} e^{-\sigma m^2} d\sigma \quad \text{diverges}$$

$$= \frac{1}{2\pi} \frac{1}{z-10} + (\text{regular @ } D=2)$$

alter quantities m, g

$$m(u) = \mu^{D-2} \left(m + \frac{\hbar}{D-2} \square + \dots \right)$$

\Rightarrow cancellation ~~non~~

σ -model

Divergences in Feynman expansion are where?

$\Phi: \mathbb{R}^D \xrightarrow{\text{or other metric}} M \sim$ has geometric data: g_{ij} (2-form, scalars)
work near $D=2$

$$S(\Phi) = \int \|\Phi^*(g_{ij})\|^2 + \int \Phi^*(\omega) + c \int \Phi^*(R) + \int \Phi^*(u)$$

Assume $g_{ij} = g_{ij}^{(0)} + \epsilon \delta g_{ij}$

divergences come from

$$-R_{ijkl} \partial^i \Phi^j \partial^k \Phi^l$$

$$\nabla^2 u - [\nabla^2 u]_{\Phi=\Phi_0}$$

$$\frac{1}{4\pi} \frac{1}{D-2} \int (-R_{ijkl}(\Phi) \partial^i \Phi^j \partial^k \Phi^l + \frac{1}{2} (\Delta_g u(\Phi) - \Delta_g u(\Phi_0))) + (\text{regular @ } D=2)$$

$$g_0 = \mu^{D-2} \left(g + \frac{\hbar}{2-D} \delta g_1 + \mathcal{O}(\hbar^2) \right) \quad @ \text{ 1-loop}$$

$$u_0 = \mu^0 \left(u + \frac{\hbar}{2-D} \delta u_1 + \mathcal{O}(\hbar^2) \right)$$

$$\Rightarrow (\delta g_1)_{ij} = R_{ij} \quad \delta u_1 = -\frac{1}{2} \Delta_g u$$

Renormalization quantities | keep g_0, u_0 "fixed" calculate
 $\mu \frac{d}{d\mu}$

$$\mu \frac{d}{d\mu} g \propto \beta_{ij}$$

$$\mu \frac{d}{d\mu} u \propto \frac{-1}{\epsilon} \Delta_g u$$

$$\frac{d}{dt} g_{ij}(t) = -\beta_{ij} + \dots$$

$$\frac{d}{dt} u(t) = \frac{-1}{\epsilon} \Delta_{g(t)} u$$

