13 April 2007
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Perturbative Gauge Theory and Arithmetic Topology

Work with Don Zagier

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3-manifolds $\longleftrightarrow$ algebraic number ~~theory~~ fields

Knots, ~~links~~ $\longleftrightarrow$ primes

⋮

$M = 3$-manifold $\longleftrightarrow$ algebraic # field $\mathbb{K}$.

$S^3 \longleftrightarrow \mathbb{Q}$

$H^n(M) \longleftrightarrow H^n_{\acute{e}t}$

links $\longleftrightarrow$ ideals in $\mathcal{O}_{\mathbb{K}}$ (e.g. $\mathcal{O}_{\mathbb{Q}} = \mathbb{Z}$)

knots $\longleftrightarrow$ prime ideals in $\mathcal{O}_{\mathbb{K}}$.

$H_{tor}(M) = \mathrm{Tor}\, H_1(M,\mathbb{Z}) \longleftrightarrow Cl(\mathbb{K})$

$H_{free}(M) = H_1(M,\mathbb{Z})/\text{torsion} \longleftrightarrow \mathcal{O}_{\mathbb{K}}^{*}/\text{roots of unity}$

finite branched coverings $\pi : M \to N$ $\longleftrightarrow$ finite extensions of $\mathbb{K} \subseteq \mathbb{L}$

Ex: S^3 has no nontrivial coverings $\longleftrightarrow$ $\mathbb{Q}$ has no nontrivial unramified extensions

Poincaré conjecture: S^3 is the only M with this property $\longleftrightarrow$?

Given M,

$$Z(M) = \int \mathcal{D}A \; e^{\frac{i}{\hbar} S_0(M,A)}$$

$$\rightsquigarrow \quad Z(M) = \exp\left(\frac{1}{\hbar} S_0 + S_1 + \hbar S_2 + \cdots\right) \qquad \hbar \to 0$$

S_n = perturbative "n-loop contributions"

Ex: S_2 = 

<u>Def</u> TQFT = QFT / metric dependence

$Z(M)$, S_n are topological invariants of M
(typically, $S_n \in \mathbb{Q}$).

<u>Def</u> "arithmetic TQFT" if $S_n \in \mathbb{K}$ for all n.

Chern-Simons theory w/ gauge group $G = SU(2)$.

$$Z(M) = \cancel{\exp\, i/4} \int \mathcal{D}A \exp\left(\frac{i}{\hbar} \int_M A\,dA + \frac{2}{3} A^3\right)$$

$$= \sum_{\substack{\text{flat} \\ \text{connections}}} \exp\left(\frac{1}{\hbar} S_0 - \frac{\delta}{2} \log \hbar + S_1 + \sum_{n=1}^{\infty} \hbar^n S_{n+1}\right)$$

$$k \to \infty$$
$$\hbar = \frac{2\pi}{k} \to 0$$

S_0 = Chern-Simons function evaluated on a flat G connection A.

$$F_A = dA + A \wedge A = 0$$

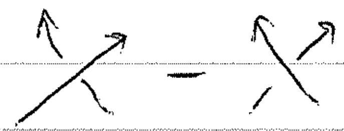
- $\delta = 3 + h^1 - h^0$, $\quad h^i = \dim H^i(M, d_A)$.
- $S_1 = \frac{1}{2} \log \frac{T(M,E)}{2}$ torsion
- $S_n(M) = \int_{M^{2n-2}} L^{3n-3}$ $\qquad L(x,y) \in \Omega^2(M_x \times M_y, \mathfrak{g} \otimes \mathfrak{g})$

Lie G.

$$d_A L(x,y) = \delta^3(x,y)$$

In SU(2) Chern-Simons theory, $S_n \in \mathbb{Q}$, $n \geq 2$.

(finite type / Vassiliev invariants)

"derivative"$(K) =$ 

$$\mathbb{Q} \subseteq \overline{\mathbb{Q}} \subseteq \mathcal{P} \subseteq \mathbb{C}.$$

periods

<u>Def</u> A <u>period</u> is a complex number whose Re and Im are absolutely converging integrals

$$\int_{D \subseteq \mathbb{R}^n} (\text{rational})$$

e.g., Π or Π^k.

Chern-Simons theory, $G = SL(2,\mathbb{C})$.

on $M = \mathbb{H}^3/\Gamma$. $\Gamma \subseteq PSL(2,\mathbb{C})$

(3D gravity = $SL(2,\mathbb{C})$ Chern-Simons)

Conj. $S_n = \int_{M^{2n-2}} L^{3n-3} \in \mathbb{K} = \mathbb{Q}(\mathrm{tr}\,\Gamma)$, $n \geq 2$

($\mathbb{K}$ depends on M)