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Kähler–Ricci Flow

(M, g) complex manifold $\qquad n = \dim M$.

$$\omega = \sum g_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$$

ω <u>Kähler</u> if $d\omega = 0$.

$$\mathrm{Ric}_g = -\underbrace{\frac{i}{\pi}\partial\bar{\partial}}_{dd^c} \log \det g \quad \text{, locally}$$

$$= -dd^c \log \det g$$

$$\mathrm{Ric}_g - \mathrm{Ric}_h = -dd^c\left(\log \det \frac{g}{h}\right)$$

$[\mathrm{Ric}_g]$ is independent of g
$\quad \shortparallel$
$c_1(M)$ $\qquad$ first Chern class

If $\mathrm{Ric}_g = Kg$ then g is a Kähler-Einstein metric

Ricci flow $\quad \partial_t g(t) = -Ric_{g(t)}$

$$g(0) = g_0$$

If g_0 is Kähler then $g(t)$ is Kähler ~~[illegible]~~, $0<t<T$.

(Bando if M is compact
Shi if M is complete, non-compact)

Important Case

M compact

$c_1(M) = K[g_0]$

$K=1$, $c_1(M)>0$, M is Fano.
$K=0$, $c_1(M)=0$, M is Calabi-Yau
$K=-1$, $c_1(M)<0$, M is canonically polarized.

$$\begin{cases} \partial_t g = -Ric_{g(t)} + Kg(t) \\ g(0) = g_0 \end{cases}$$

(Normalized Kähler-Ricci flow)

$$\partial_t [g] = [-Ric_{g(t)} + Kg(t)] = 0$$

$$\Rightarrow [g(t)] = [g(0)]$$

$$g(t) = g_0 + dd^c u.$$

$c_1(M) = K[g_0]$.

$\Rightarrow \exists f$ such that $Ric_{g_0} = Kg_0 - dd^c f$

$$\partial_t(dd^c u) = dd^c \log\det g(t)/g_0 + dd^c f + K dd^c u.$$

NKRF $\Rightarrow$ (*)
$$\begin{cases} \partial_t u = \log\det\left(\cancel{g(t)}\ {}^{g_0 + dd^c u}/g_0\right) + f + Ku. \\ u(0) = 0. \end{cases}$$

Existence

① Short time existence

Hamilton: (M compact)

Shi (M complete, non compact, bounded sectional curvature)

② Long time existence

Cao if M compact, $c_1(M) = K[g_0]$

Let $V = \partial_t u$

$$\partial_t (*) \Rightarrow \partial_t V = \Delta_{g(t)} V + KV$$

Max Princ. + Yau's estimates $\Rightarrow$ long time existence.

$$|V| \leq Ce^{Kt}$$

Shi complete, noncompact, bounded sectional curvature
+ nonnegative sectional curvature
$$+ \frac{1}{\mathrm{vol}(B_x(r))} \int_{B_x(r)} R\, dv \leq \frac{C}{(1+r)^p}$$
$x \in M$ fixed, $C, p > 0$.

Applications

1) M compact $c_1(M) = k[g_0]$

Hamilton (1988), Chow (1992):

if $n=1$ (M = Riemann surface)

$\Rightarrow g_\infty = \lim_{t\to\infty} g(t)$ is a smooth K-E.

2) • **Cao** ('85) if $k = 0, -1$

$\Rightarrow g_\infty = \lim_{t\to\infty} g(t)$ is smooth K-E.

(Aubin-Yau '76)

Mori ('79) + Siu-Yau ('80)

M compact, positive holo bisectional curvature then $M \cong \mathbb{P}^n_{\mathbb{C}}$.

$n=3$ (Bando), $n \geq 3$ (Mok)

non-negative bisect. curvature is preserved by Ricci Kähler flow

if bisectional curvature of g_0 is >0 at one point
then $g(t)$ has positive sectional (Ricci) curvature everywhere.

Bando $n=3$; if $\pi_1(M)=1 \Rightarrow M \cong \mathbb{P}^a \times \mathbb{P}^b \times \mathbb{P}^c$
$a+b+c=3$
or $M \cong Q \subseteq \mathbb{P}^4$ quadric

Mok $\tilde{M} \cong \mathbb{P}^{a_1} \times \mathbb{P}^{a_2} \times \cdots \times M_1 \times M_2 \times \cdots \times \mathbb{C}^k$

where M_i are hermitian symmetric manifolds.

Idea
{Berger} (1955)
enough to show $\exists\, S \subseteq \mathbb{P}(T_M)$
proper and invariant under holonomy.

After deforming g_0 with R.K.F.

$\longrightarrow Ric_{g(t)} > 0$.

Mori $\Rightarrow \exists\, f : \mathbb{P}^1 \to M$.
$S = \{$ direction of $f(\mathbb{P}^1)$ with minimal degree $\}$

If $S = \mathbb{P}(T_M)$ then $M \cong \mathbb{P}^n$.

Otherwise S is proper.

Sectional curvature $> 0 \Rightarrow S$ is invariant $\square$.

Chen-Tian ('06)

If bisectional curvature > 0 then g_∞ is Fubini-Study metric on $\mathbb{P}^n$.

Conjecture

$c_1(M) = [g_0]$

Assume $\exists$ KE metric. Is g_∞ a KE?

Perelman

if $c_1(M) = [g_0)$ $\exists$ $C = C(g_0)$ s.t.

$$|R_{g(t)}| < C$$

$$|\text{diameter of } g(t)| < C$$

2) M is non-compact

Conj (Yau) M complete, non-compact with bisectional curvature > 0 then $M = \mathbb{C}^n$.

$\Rightarrow \pi_1(M) = 1$

Chen-Zhu (2005)

Previous assumption, + bounded bisectional curvature, +

$$\frac{1}{\mathrm{vol}(B_x(r))} \int_{B_x(r)} R \leq \frac{\varepsilon(r)}{1+r^2}$$

$$\varepsilon(r) \xrightarrow[r \to \infty]{} 0$$

$$\Rightarrow \tilde{M} \cong \mathbb{C}^n.$$

K_X nef, long time existence (C, LeNave; Tian, Zhang)

$g_\infty = ?$

if the canonical model has orbifold singularities then g_∞ induces a metric on the canonical model.

if $K_X \cdot C < 0$ there is a finite time s.t. $g(T)$ singular along C