Nora Ganter
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Elliptic cohomology, generalized Moonshine, some connections to physics

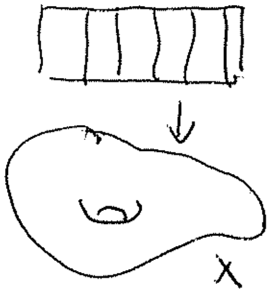
generalized cohomology theories

1) Singular cohomology

$H^*(X)$, $H^*(pt) = \mathbb{Z}$

X

2) topolog K-theory



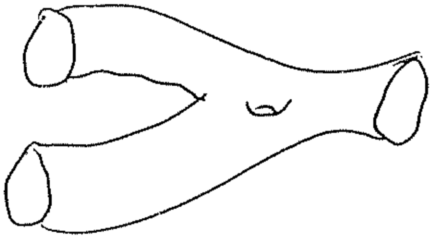
vbdl/X

X

$K(X) = (\mathrm{Vect}_{\mathbb{C}}(X), \oplus)^{gp}$

3) Complex cobordism $MU^*(X)$

$MU^*(pt) =$ cx cobordism ring



X   X'

Quillen's thm:

Def A complex genus is a map of rings $\Phi : MU^*(pt) \to R$

<u>Thm</u> (Quillen)

$\{$ cx genera w/ values in $R\}$ $\overset{1-1}{\longleftrightarrow}$ $\{$ formal group laws $/R\}$

<u>Examples</u>

1) <u>FGL</u> | | <u>Genus</u>

$\mathbb{G}_a$, additive $F(x,y) = x+y$ $\qquad$ $M \mapsto \begin{cases} \# M & \text{if } \dim M = 0 \\ 0 & \text{if } \dim M > 0 \end{cases}$

natural transf. $MU^*(-) \to H^*(-)$

2) $\mathbb{G}_m$, multip. $F(x,y) = x+y-xy$ $\qquad$ $M \mapsto Td(M)$

Todd genus or "cplx Euler char."

Conner Floyd map $MU^*(-) \to K^0(-)$

3) C/R elliptic curve

$\hat{C}_0$ formal grp

$$E(CP^\infty) = E^*(pt)[[x]]$$

<u>Def</u>: $\boxed{\text{An}}$ elliptic genus is a genus ϕ_{ell} $MU^*(pt) \to R$ that classifies a formal grp of the form $\hat{C}_0$

<u>Def</u> An elliptic cohomology theory is a generalized multiplicat. cohomology theory Ell, together w/ an elliptic curve $C/Ell^*(pt)$

and a natural transformation $MU(-) \longrightarrow Ell(-)$ realizing the elliptic genus defined by C_0^1.

Ell's have been constructed using hard homotopy theoretic techniques.

G - finite group $Ell_G(pt)$: compare w/ $K_G(pt) = R(G)$

Hopkin - Kuhn - Ravenel

$E_2(BG) \ni \chi$ are "2 class fct"

$$\chi(g,h) = \chi(s^{-1}gs, s^{-1}hs)$$
$$gh = hg$$

Orbifold elliptic genera

$$MU^*(BG) \xrightarrow{\phi_{ell}} Ell^*(BG) \xrightarrow{(*)} Ell^*(pt)$$

global notion orbifold $M//G \cong \mathcal{M}/H$

$(*)$ $\chi \longmapsto \frac{1}{|G|} \sum_{gh=hg} \chi(g,h)$

$$\phi_{ell,orb}(M) = \frac{1}{|G|} \sum_{gh=hg} \phi_{ell,G}(M)(g,h)$$

$$M//G \longrightarrow \bullet//G \longrightarrow \bullet//1$$

$$M \circlearrowleft G \longrightarrow pt \circlearrowleft G$$

$$G \longrightarrow 1$$

$$\text{Borel}(M//G) \longrightarrow BG \to pt$$

in $\mathcal{S}_{K(2)}$ localization of the homotopy theory, there is a notion of duality $\mathcal{D}(K(2))$ and BG becomes self-dual

$$\mathcal{D}_{K(2)}(BG_+ \longrightarrow B\cancel{G}1_+) : S^0 \longrightarrow BG_+$$

BG_+: disjoint base pt

$$\mathcal{D}_{K(2)}(\underline{\qquad}) = \underline{\qquad}$$

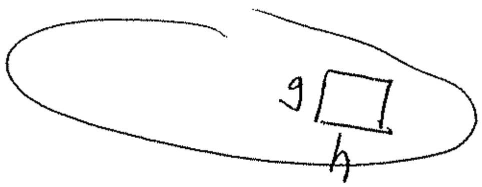
$$\phi_G \quad \chi \mapsto \frac{1}{|G|} \sum_{gh=hg} \chi(g,h)$$

Generalized Moonshine,
Ginzburg, Kapranov — Vasserot

axioms for Ell_G

P
$\downarrow$


— principle G — bdls over ellipt. curve



monodromies

Conj

Nation: There is a section "$\mathcal{F}$" over the moduli stack of principle [illegible] M over cx ellipt. curves

- once you fix g and h the function $\mathcal{F}(g, h, \tau)$ has invariance grp Γ

Γ'

$\Gamma \subset \Gamma' \subset Sl_2(\mathbb{Z})$

finite index

$\mathfrak{h}/\Gamma$

genus zero

replicability + Hecke operators

Classical Moonshine

$$j_{\langle g \rangle}(\tau) = F(1, g; \tau)$$

$$j_{\langle 1 \rangle} \Leftarrow F(1,1) = j - 744$$

$j_{\langle g \rangle}$ – class Moonshine fcn,

$$T_n(j_{\langle g \rangle})(\tau) := \frac{1}{n} \sum_{\substack{ad=n \\ 0 \le b < d}} j_{\langle g^a \rangle}\left(\frac{a\tau + b}{d}\right)$$

nth Hecke operator a^{th} Adams op

for $j_{\langle 1 \rangle}$: T_n is Hecke operator on j-fct.

$$T_n(j_{\langle g \rangle})(\tau) = \Phi_n(j_{\langle g \rangle})$$

$$= q^{-n} + b_1 q + \dots$$

$$f = q^{-1} + a_1 q + \dots$$

$$f^{(a)}$$

<u>Thm</u>: $T_n(f)(g, h; \tau) = \frac{1}{n} \sum_{\substack{ad=n \\ 0 \le b < d}} f\left(g^d, g^b h^a; \frac{a\tau + b}{d}\right)$

$\boxed{\text{Tuite} \rightsquigarrow \text{genus zero}}$