Matthew Ando: Equivariant elliptic cohomology and fibered WZW-models

mfld M, spin, $\not{D}$ = Dirac operator

tangent bdl TM,

$\not{D} : \Gamma(\Delta^+) \longrightarrow \Gamma(\Delta^-)$

$\mathrm{ind}(\not{D}) \in \mathbb{Z}$

more general:

$$\begin{array}{ccc} K(M) & 1 & V \\ \Big\downarrow{\scriptstyle \mathrm{ind}} & \Big\downarrow & \Big\downarrow \\ K(pt) \in \mathbb{Z} & \mathrm{ind}(\not{D}) & \mathrm{ind}(\not{D}; V) \end{array}$$

$\not{D} \otimes V : \Gamma(\Delta^+ \otimes V) \longrightarrow \Gamma(\Delta^- \otimes V)$

$\mathrm{ind}(\not{D}; V) \in \mathbb{Z}$

$$S_t(\underset{\substack{\uparrow \\ \text{cplx}}}{V}) = \sum_{n \geq 0} t^n \, \mathrm{Sym}^n(V) \in K(X)[[t]]$$

$$W(M) = \mathrm{ind}\Big(\not{D}, \bigotimes_{n \geq 1} S_{q^n}\big(T_{\mathbb{C}} - \underset{\substack{\uparrow \\ \text{rank } T}}{d}\, \underset{\substack{\uparrow \\ \text{trivial bdl}}}{\underline{\varepsilon}}\big)\Big) \in K(pt)[[q]]$$

$$\frac{1}{1-r} = 1 + r + r^2 + \ldots$$

$$\frac{1}{\Lambda_{-t} V} = S_t(V) \qquad S_t(V)^{-1} = \Lambda_{-t} V$$

$W(M)$ is called „Witten genus" of M.

$$\underbrace{(1 + q\, T_{\mathbb{C}} + q^2 S^2 T_{\mathbb{C}} + \ldots)}_{n=1} \underbrace{(1 + q^2 T_{\mathbb{C}} + \ldots}_{n=2}$$

$$q = e^{2\pi i \tau}$$

$$\begin{tikzcd}
BSU \arrow[r, "\cong"', "H^4"'] \arrow[d] \arrow[rr, bend left, "c_2"] & BSpin \arrow[r, "\lambda", "c_2"'] \arrow[d] & K(\mathbb{Z};4) \arrow[d, "2"] \\
BU \arrow[r] & BSO \arrow[r, "p_1"'] & K(\mathbb{Z};4)
\end{tikzcd}$$

$$S^3 = Spin(3) = SU(2)$$

<u>digression</u>:

$$\left\{ \begin{matrix} \text{cplx line} \\ \text{bundles} \end{matrix} \right\} / X \longleftrightarrow H^2(X;\mathbb{Z})$$

$$H^2(X;\mathbb{Z}) \; \| \; [X \to K(\mathbb{Z};2)]$$

$$K(\mathbb{Z};2) = \mathbb{C}P^\infty = BU(1)$$

$$\begin{tikzcd}
f^*L \arrow[r] \arrow[d] & L \arrow[d] \\
X \arrow[r, "f"'] & \mathbb{C}P^N
\end{tikzcd}$$

$$\pi_i \, K(\mathbb{Z};2) = 0$$

$$\pi_2 \, K(\mathbb{Z};2) = \mathbb{Z}$$

$$\begin{tikzcd}
S^1 \arrow[d] \\
S^{2N+1} \arrow[d] \\
\mathbb{C}P^N
\end{tikzcd}$$

$\mathbb{C}P^N$ = lines in $\mathbb{C}^{N+1}$

real case

$$S^0 \longrightarrow S^2 \longrightarrow \mathbb{R}P^2$$

$$S^0 \longrightarrow S^n \longrightarrow \mathbb{R}P^n$$

$$S^1 \to \underset{*}{\overset{\wr}{S^\infty}} \to \mathbb{C}P^\infty \qquad \pi_{i+1}(\mathbb{C}P^\infty) = \pi_i(S^1)$$

$$\begin{tikzcd} L \arrow[d] & & \mathbb{C}^* \arrow[r] & P \arrow[d] \\ X & \rightsquigarrow & & X \end{tikzcd}$$

cpct Lie grp:

$$G \longrightarrow BG$$

$$\begin{tikzcd} G \arrow[r] & P \arrow[d] & & P \cong f^*\xi \arrow[r] \arrow[d] & \xi \arrow[d] & G \arrow[l] \\ & X & \rightsquigarrow & X \arrow[r, "f"'] & BG & \end{tikzcd}$$

classify space for principal G-bdls

$$\begin{tikzcd} \text{Spin bundles}/X \arrow[r, leftrightarrow] \arrow[d, rightsquigarrow, "p_1"] & (X \to BSpin) \arrow[d, rightsquigarrow] \\ H^4(X;\mathbb{Z}) \arrow[r, leftrightarrow] & (X \to K(\mathbb{Z};4)) \end{tikzcd}$$

line bdls: $c_1 \leftrightsquigarrow H^2(\mathbb{C}P^\infty;\mathbb{Z})$

$$\updownarrow$$

$$\mathbb{C}P^\infty \xrightarrow[id]{\cdot} K(\mathbb{Z};2) = \mathbb{C}P^\infty$$

vct - bdls:

$$\mathbb{Z}[c_1, c_2, \dots] \cong H^*(BU(n); \mathbb{Z})$$

$\deg c_i = 2i$, c_i: chern-classes

$$BSU \searrow BU \xrightarrow{c_2} K(\mathbb{Z}; 4)$$

$T \to M$ spin $\rightsquigarrow$ $w(T) \in \mathbb{Z}[[q]]$

If $c_2(T) = 0$ then $w(T)$ is the q expansion of a modular form

(Proofs: 1) Zagier,
2) Witten: $\Delta^{\pm} \otimes \bigotimes_{n \geq 1} S_{q^n}(T^{\mathbb{C}} - d\,\varepsilon)$
or $\Delta^{\pm} \otimes \bigotimes_{n \geq 1} S_{q^n}(T^{\mathbb{C}})$

closed strings: $\Sigma \longrightarrow M$

$W(M) = \langle \mathcal{O} \rangle$ 1-loop-amplitude

$a \;\rule{2em}{0.4pt}\; b$ $\quad \langle b | e^{tH} | a \rangle$

$$\sum_a \langle a | e^{tH} | a \rangle = \underset{S^1}{\mathrm{Str}}\, e^{tH}$$

QFT is only well-defined if $c_2(T) = 0$

$$\begin{tikzcd}
\mathbb{C}P^\infty \times \mathbb{C}P^\infty \times \mathbb{C}P^\infty \arrow[r] & BU\langle 6\rangle \arrow[r] \arrow[d, "(L_1-1)(L_2-1)"] & BO\langle 8\rangle = BString \arrow[d] & \\
CP^\infty \times CP^\infty \arrow[r, "J"] & BSU \arrow[r] \arrow[d] & BSpin \arrow[r, "c_2"] \arrow[d] & K(\mathbb{Z};4) \\
\mathbb{C}P^\infty \arrow[r] & BU \arrow[r] & BSO \arrow[r, "w_2"] \arrow[d] & K(\mathbb{Z}/2;2) \\
& & BO \arrow[r, "w_1"] & K(\mathbb{Z}/2,1)
\end{tikzcd}$$

line bdls: $\underset{x}{L_1}\underset{y}{L_2}\underset{z}{L_3} - L_1L_2 - L_1L_3 - L_2L_3 + L_1 + L_2 + L_3 - 1$

$$\frac{f(x+y+z)\,f(x)\,f(y)\,f(z)}{f(x+y)\,f(x+z)\,f(y+z)}$$

$$\frac{\sigma(x)\quad\sigma(x+y+z)}{\sigma(x+y)\,\sigma(x+z)}$$

divisor $\quad (0) + (-y+z) - (-y) - (-z)$

$$Sq^n\big(\underset{\underset{L+\bar{L}}{\parallel}}{L^{\mathbb{C}}} - d\varepsilon\big)$$

$$(L^{1/2}-L^{-1/2})\prod \frac{(1-q^nL)(1-q^n\bar{L})}{(1-q^n)^2} = \sigma(L,\underset{\underset{e^{2\pi i\tau}}{\parallel}}{q})$$

so far: "Witten genus is modular" ↔ String bdls are oriented in Ell

"Witten genus is rigid"

$$S^1 \circlearrowright \begin{matrix} T \\ \downarrow \\ X \end{matrix} \qquad W_{S^1}(M) = \mathrm{ind}_{S^1}\Big(\not{D};\ \bigotimes_{n\geq 1} S_{q^n}(T^{\mathbb{C}} - d\varepsilon)\Big) \in R[S^1]\,[\![q]\!]$$

if $c_2^{S_1}(T/X) = 0$ then $W_{S^1}(M) = W(M)$

a priori $W_{S^1}(M) = \sum_{-\infty \ll n \ll \infty} a_n(q)\, \lambda^n$

all trivial but a_0

$\rightsquigarrow$ TMF $\longrightarrow$ MF

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(compare w/ Atiyah, Hirzebruch.

if M S^1-~~action~~ $\Rightarrow$ $\hat{A}_{S^1}(M) = 0$)

<u>Proof</u>: Taubes, Bott-Taubes, K. Liu, Rosu (Ochanine), Ando-Basterra...

S^1-equiv. string bdls are oriented in S^1-equiv elliptic cohomology

$$\begin{tikzcd} z \in \mathbb{T} \arrow[d] & \\ \mathbb{T}_{Rot} \ltimes \widetilde{L\,Spin(2d)} \arrow[r, "\text{Level } K"'] \arrow[d] & Aut(H) \\ \mathbb{T}_{Rot} \ltimes L\,Spin\ (2d) & \end{tikzcd}$$

means $z \mapsto z^k$

$$\begin{tikzcd} \mathbb{T}_{rot} \times T \times T_{cent} \times \mathbb{T}_{rot} \arrow[d] \\ T \end{tikzcd}$$

$$H = \bigoplus_{n \in \mathbb{Z}} q^n \, H_n$$

$\circlearrowright$ T

restrict to $\bigoplus_{n \geq 0} q^n \; H_n$

character

$$\chi = \sum \underbrace{a_n(\lambda)}_{a_n = (z_1, \ldots, z_d)} q^n$$

$$\lambda \in T = (z_1, \ldots, z_d)$$

$$Ell_{Spin(2d)}(pt)$$

$$\mathbin{\rotatebox{90}{=}} \left(\check{T} \otimes (\mathbb{C}^\times / q^{\mathbb{Z}}) \big/ \text{Weyl grp} \right)$$

$$\underset{\text{Loojenga}}{\cong} \quad \mathbb{P}(g_0, \ldots, g_d) \quad \text{weighted proj space}$$

we have a line bdl $\mathcal{L}$ on it

$$\chi \in \Gamma \left(\overset{\mathcal{L}^{\otimes k}}{\downarrow} \; \check{T} \otimes (\mathbb{C}^* / q^{\mathbb{Z}}) \big/ \text{Weyl} \right) \cong Rep_K(L\,Spin(2d))$$

$L\,Spin(2d)$ has 1 irred. repr. for level 1 "Basic rep."

$$\chi(z_1, \dots, z_d) = \prod_{i=1}^{d} \sigma(z_i)$$

$$\begin{array}{l} G \searrow P \\ \quad\ \downarrow \\ C = S^1 \times S^1 = \hom(\underset{\mathbb{Z}\times\mathbb{Z}}{\pi_1 C}, G)/\text{conj} \end{array}$$

S^1-equivariant cohomology:

E: S^1-spaces $\to$ sheaves f. $\mathcal{O}_C$-alg

$\uparrow$ ell. curve

$E(pt) = \mathcal{O}$

$E(X)^\wedge_0 = E^{non}(ES^1 \times_{S^1} X)$

$$\begin{array}{ccc} \text{Dirac} & E(T) & \\ \downarrow & \downarrow & P - \text{Thom const.} \\ \text{index} & E(pt) & \end{array}$$

$$\begin{array}{ccc} W & \in E^{non}(ES^1 \times_{S^1} T\mathbb{Z}) \\ \downarrow & \downarrow \\ W_{S^1}(M) & E^{non}(BS^1) \end{array}$$

Q : does there exist B such that

$$\begin{tikzcd} B \in \Gamma(E(T)) \arrow[r, mapsto] \arrow[d, mapsto] & W \in E^{hol}(ES^1 \times_{S^1} T) \arrow[d, mapsto] \\ A(E(pt)) \arrow[r, mapsto, "f"] & W_{S^1}(M) \quad E^{hol}(BS^1) \end{tikzcd}$$

$\Gamma(\mathcal{O}_C) = \mathbb{C}$

it exists if $c_2^{S^1}(T) = 0$

$\rightsquigarrow$ T , V spin bdls

if $c_2^{S^1}(T) = c_2^{S^1}(V)$ then

$W_{S^1}(M; V)$ is rigid

$T - V$ virtual string bdl

$$\begin{tikzcd} & G \arrow[d] \\ T \arrow[dr] & P \arrow[d] \\ & M \end{tikzcd}$$

$G \leftarrow$ simple, s.c. Lie grp

ν rep. of LG of level K

$W(M; \nu)$ rigid

if $c_2(P/_M) = c_2(T)$

$$\begin{tikzcd} & \mathcal{L} \arrow[d] \\ \operatorname{spec} E(X) \arrow[r] & M_{spin(2d)} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{M}^{\otimes k} \arrow[d] & & \mathcal{M}^{\otimes k} \arrow[d] & & \mathcal{L} \arrow[dll] & & \mathcal{L} \arrow[d] \\ M_G \arrow[u, bend right, "\nu"'] & & \operatorname{spec} E(X) \arrow[ll] \arrow[u, bend right] \arrow[urr, bend right, "\sigma"'] \arrow[rrrr] & & & & M_{spin(2d)} \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

$$\Theta \otimes \sigma^{-1} \in \Gamma(\mathcal{L} \otimes \mathcal{M}^{\otimes -k}) \cong \Gamma(\alpha_c)$$

$$\begin{tikzcd} G \arrow[dr] & & \\ & P \arrow[d] & \Sigma \arrow[l] \\ & M & \end{tikzcd}$$

Lie alg of currents $\cong \mathrm{Lie}(\widetilde{LG})$

$$c_2(M) = K\, c_2(P)$$

$$\begin{tikzcd} \Sigma \arrow[r, "f"] \arrow[d, phantom, "\cap"] & G \\ B \arrow[ur] & \end{tikzcd} \qquad \nu \in \Omega^3$$

w/ $\partial B = \Sigma$

$$S_{WZW} = \int_B f^* \nu$$ closed 3-form

So Hilbert space of states is a Rep of LG