

Matthew Ando: Equivariant elliptic cohomology and fibered WZW-models ①

mfld  $M$ , spin,  $\not{D}$  = Dirac operator

tangent bdl.  $TM$ ,

$$\not{D}: \Gamma(\Delta^+) \rightarrow \Gamma(\Delta^-)$$

$$\text{ind}(\not{D}) \in \mathbb{Z}$$

more general:

$$\begin{array}{ccc} K(M) & & V \\ \text{ind} \downarrow & \begin{array}{c} \mathbb{1} \\ \downarrow \\ \text{ind}(\not{D}) \end{array} & \downarrow \\ K(pt) \in \mathbb{Z} & & \text{ind}(\not{D}; V) \end{array}$$

$$\not{D} \otimes V: \Gamma(\Delta^+ \otimes V) \rightarrow \Gamma(\Delta^- \otimes V)$$

$$\text{ind}(\not{D}; V) \in \mathbb{Z}$$

$$S_t(V) = \sum_{n \geq 0} t^n \text{Sym}^n(V) \in K(x)[[t]]$$

$$W(M) = \text{ind}(\not{D}; \bigotimes_{n \geq 1} S_{q^n}(\underbrace{T_x}_{\text{rank } T} - \underbrace{d_x}_{\text{trivial bdl}})) \in K(pt)[[q]]$$

$$\frac{1}{1-r} = 1 + r + r^2 + \dots$$

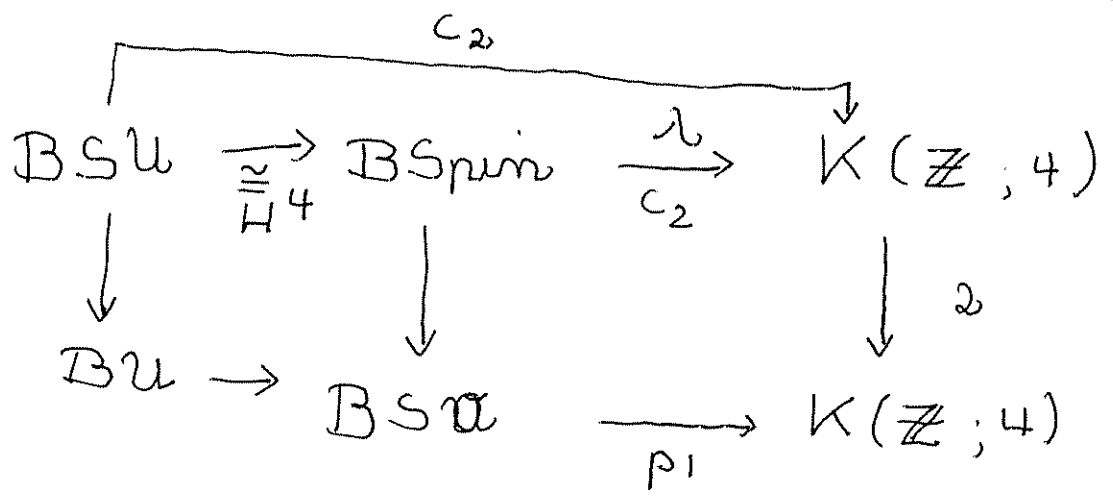
$$\frac{1}{\Lambda_{-t} V} = S_t(V) \quad S_t(V)^{-1} = \Lambda_{-t} V$$

$W(M)$  is called "Witten genus" of  $M$ .

$$(1 + q T_x + \sum_{n=1} q^{2n} S^2 T_x + \dots) (1 + \sum_{n=2} q^{2n} \pi_x + \dots)$$

$$q = e^{2\pi i t}$$

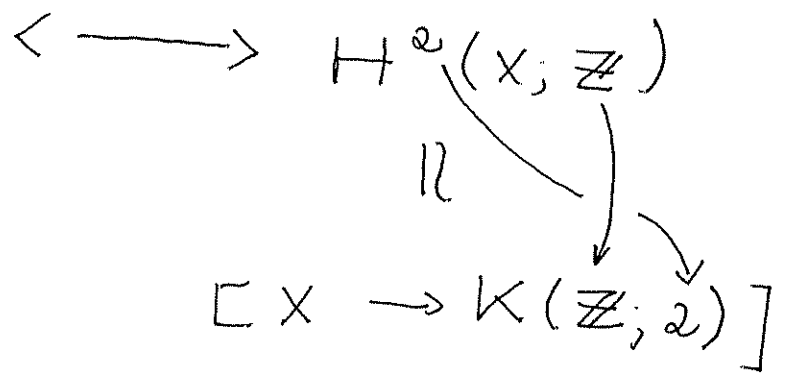
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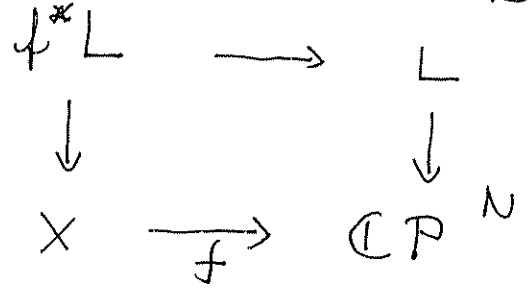
$$S^3 = Spin(3) = su(2)$$

digression:

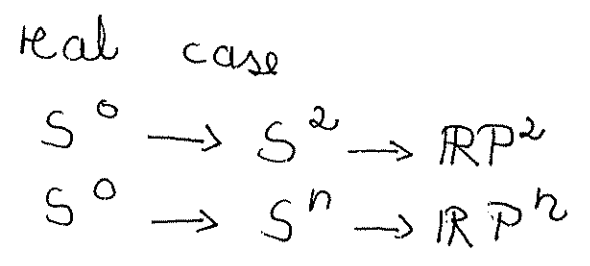
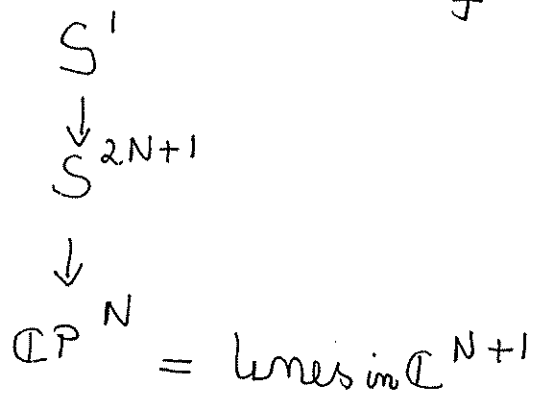
{ cplx line bundles }  
X



$$K(\mathbb{Z}; 2) = \mathbb{C}P^\infty = BU(1)$$

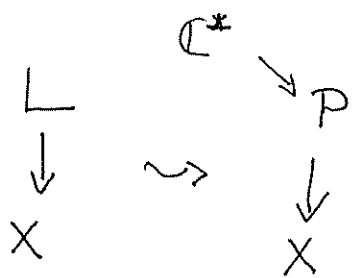


$$\begin{array}{l}
 \pi_1 K(\mathbb{Z}; 2) = 0 \\
 \pi_2 K(\mathbb{Z}; 2) = \mathbb{Z}
 \end{array}$$

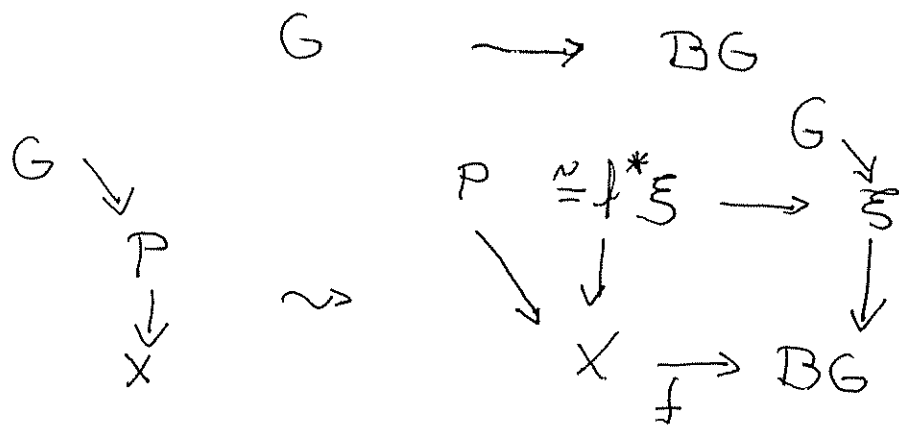


$$S^1 \rightarrow S^{\infty} \rightarrow \mathbb{C}P^{\infty}$$

$$\pi_{i+1}(\mathbb{C}P^{\infty}) = \pi_i(S^1) \quad (3)$$



cpct lie grm:



classify space for principal G-bdls

$$\text{Spin bundles over } X \longleftrightarrow (X \rightarrow BSpin)$$

$$\left. \begin{array}{c} \downarrow \\ \{ P_1 \} \\ \downarrow \end{array} \right\} H^4(X; \mathbb{Z}) \longleftrightarrow (X \rightarrow K(\mathbb{Z}; 4))$$

line bdl:  $C_1 \rightsquigarrow H^2(\mathbb{C}P^{\infty}; \mathbb{Z})$

$$\begin{array}{c}
 \mathbb{C}P^{\infty} \xrightarrow{\text{incl}} K(\mathbb{Z}; 2) = \mathbb{C}P^{\infty}
 \end{array}$$

vec - bds:

(4)

$$\mathbb{Z} [c_1, c_2, \dots] \cong H^*(BU(n); \mathbb{Z})$$

deg  $c_i = 2i$ ,  $c_i$ : Chern-classes

$$BSU \rightarrow BU \xrightarrow{c_2} K(\mathbb{Z}; 4)$$

$$T \rightarrow M \text{ spin} \rightsquigarrow w(T) \in \mathbb{Z} [\![q]\!] ]$$

If  $c_2(T) = 0$  then  $w(T)$  is the  $q$  expansion of a modular form

(Proofs: 1) Zagier,

2) Witten:  $\Delta^\pm \otimes \bigotimes_{n \geq 1} S_{q^n}(T^e - d\epsilon)$

or  $\Delta^\pm \otimes \bigotimes_{n \geq 1} S_{q^n}(T^e)$

closed strings:

$$\Sigma \longrightarrow M$$

$$W(M) = \langle \emptyset \rangle \text{ 1-loop-amplitude}$$

$$\begin{array}{c} \text{~~~~~} \\ a \quad b \end{array} \langle b | e^{tH} | a \rangle$$

$$\sum_a \langle a | e^{tH} | a \rangle = \text{Str}_S e^{tH}$$

QFT is only well-defined if  $c_2(T) = 0$

$$\begin{matrix} \mathbb{C}P^\infty \\ \mathbb{C}P^\infty \times \mathbb{C}P^\infty \\ \mathbb{C}P^\infty \times \mathbb{C}P^\infty \end{matrix} \rightarrow BU\langle 6 \rangle \rightarrow BO\langle 8 \rangle = BString$$

$$\begin{matrix} (L_1, -1) & (L_2, -1) \\ \downarrow & \downarrow \\ \mathbb{C}P^\infty \times \mathbb{C}P^\infty & \xrightarrow{j} BSU \end{matrix} \rightarrow \begin{matrix} \downarrow \\ BSpin \end{matrix} \xrightarrow{c_2} K(\mathbb{Z}, 4)$$

$$\mathbb{C}P^\infty \rightarrow BU \rightarrow BSO \xrightarrow{w_2} K(\mathbb{Z}/2, 2)$$

$$\downarrow \\ BO \xrightarrow{w_1} K(\mathbb{Z}/2, 1)$$

line bdds:  $L_1, L_2, L_3 - L_1 L_2 - L_1 L_3 - L_2 L_3 + L_1 + L_2 + L_3 - 1$

$$\frac{f(x+y+z) f(x) f(y) f(z)}{f(x+y) f(x+z) f(y+z)}$$

$$\frac{\sigma(x) \sigma(x+y+z)}{\sigma(x+y) \sigma(x+z)}$$

divisor  $(0) + (-y+z) - (-y) - (-z)$

$$Sq^n \left( \frac{L^{\mathbb{C}} - d\varepsilon}{L'' + \bar{L}} \right)$$

$$(L^{1/2} - L^{-1/2}) \prod \frac{(1 - q^n L)(1 - q^n \bar{L})}{(1 - q^n)^2} = \sigma(L, q) e^{2\pi i \tau}$$

so far: "Witten genus is modular" ⑥  
 "Witten genus is rigid" String  
bds are  
oriented  
in EW

$$S^1 \begin{array}{c} \circlearrowleft \\ \downarrow \\ X \end{array} \quad W_{S^1}(M) = \text{ind}_{S^1}(\not{D}; \bigotimes_{n \geq 1} S_{q^n}(T^c - dE))$$

$\in R[S^1] \quad [q]$

$\Downarrow$   $c_2^{S^1}(T/X) = 0$  then  $W_{S^1}(M) = W(M)$

a priori  $W_{S^1}(M) = \sum_{-\infty \ll n \ll \infty} a_n(q) \lambda^n$   
 all trivial but  $a_0$

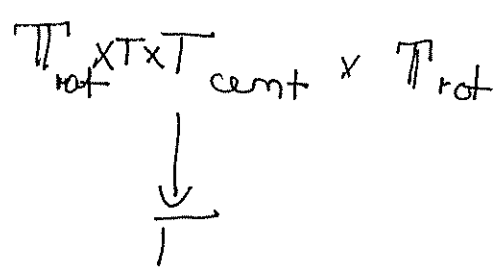
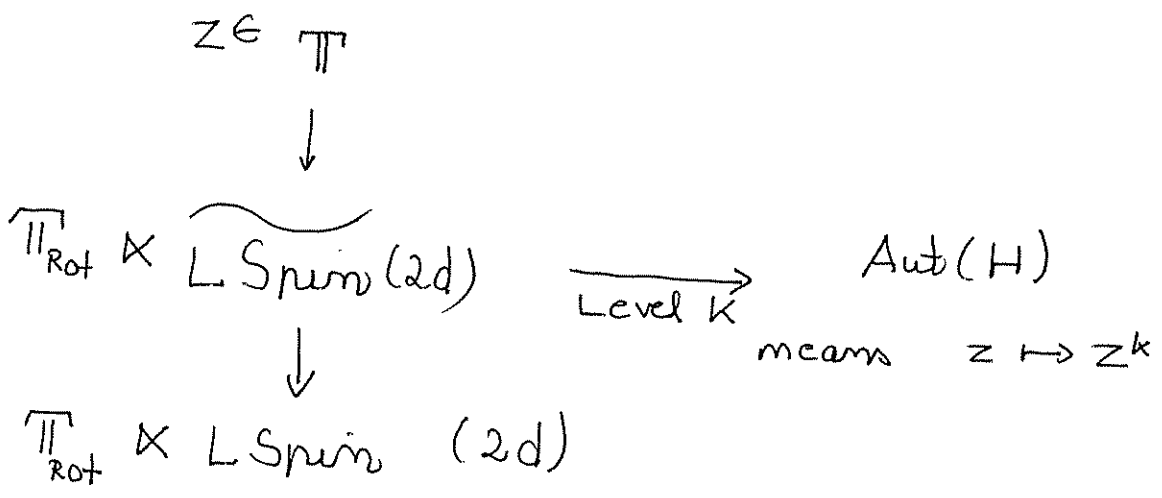
$\leadsto$  TMF  $\rightarrow$  MF  
 Hopkins Beijing ICM

(compare w/ Atiyah, Hirzebruch.

$\Downarrow$   $M$   $S^1$ -action  $\Rightarrow \hat{A}_{S^1}(M) = 0$ )

Proof: Taubes, Bott-Taubes, K. Liu,  
 Rosu (Ochanine), Ando-Barr...

$S^1$ -equiv. string bds are oriented in  
 $S^1$ -equiv elliptic cohomology



$$H = \bigoplus_{h \in \mathbb{Z}} q^h H_h$$

↑

T

restrict to  $\bigoplus_{n \geq 0} q^n H_n$

character

$$\chi = \sum_{a_n = (z_1, \dots, z_d)} \underbrace{a_n(\lambda)} q^{a_n}$$

$Ell_{Spin(2d)}(pt)$

$\lambda \in T = (z_1, \dots, z_d)$

$$\cong \left( \frac{\mathbb{T} \otimes (\mathbb{C}^* / q^{\mathbb{Z}})}{\text{Weyl group}} \right)$$

Lasjonga  $\cong$

$\mathbb{P}(g_0, \dots, g_d)$  weighted proj space

we have a line bundle  $\mathcal{L}$  on it

$$\chi \in \Gamma \left( \frac{\mathbb{T} \otimes (\mathbb{C}^* / q^{\mathbb{Z}})}{\text{Weyl}} \right) \cong \text{Rep}_k(LSpin(2d))$$

$LSpin(2d)$  has 1 irred. repr. for  $\textcircled{8}$   
 level 1 "Basic rep."

$$\chi(z_1, \dots, z_d) = \prod_{i=1}^d \sigma(z_i)$$

$$\begin{array}{c} G \\ \downarrow \\ P \\ \downarrow \\ C = S^1 \times S^1 = \text{hom}(\underbrace{\pi_1 C_{\mathbb{Z} \times \mathbb{Z}}}_{\mathbb{Z} \times \mathbb{Z}}, G) / \text{conj} \end{array}$$

$S^1$ -equivariant cohomology:

$E: S^1\text{-spaces} \rightarrow \text{sheaves of } \mathcal{O}_C\text{-alg}$

$E(\text{pt}) = \mathcal{O}_{\text{ell. curve}}$

$E(X)^\wedge = E^{\text{non}}(ES^1 \times_{S^1} X)$

Dirac  
↓  
index

$E(T)$   
↓  
 $E(\text{pt})$

$P$ -Thom const.

$W_{S^1}(M)$

$E^{\text{non}}(BS^1)$



Q: does there exist  $B$  such that

(9)

$$B \in \Gamma(E(T)) \longrightarrow W \in E^{\text{hom}}(ES' \times_{S'} T)$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \Gamma(E(\text{pt})) & \xrightarrow{f} & W_{S'}(M) \in E^{\text{hom}}(BS') \\ \Gamma(\mathcal{O}_C) = \mathbb{C} & & \end{array}$$

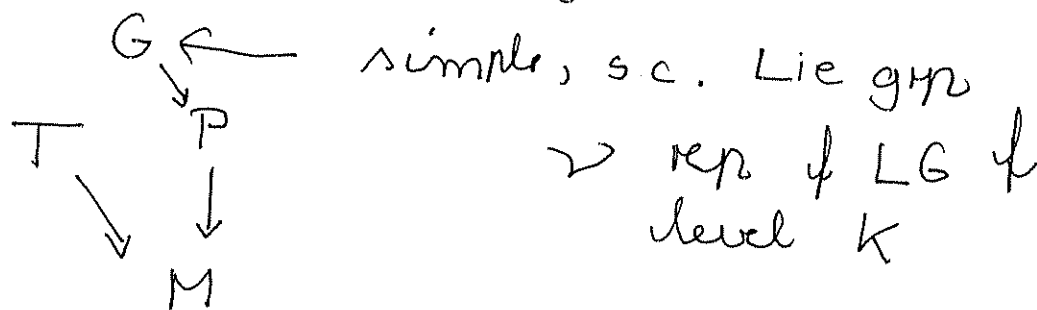
it exists  $\iff c_2^{S'}(T) = 0$

$\leadsto$   $T, V$  spin bdl

$\iff c_2^{S'}(T) = c_2^{S'}(V)$  then

$W_{S'}(M; V)$  is rigid

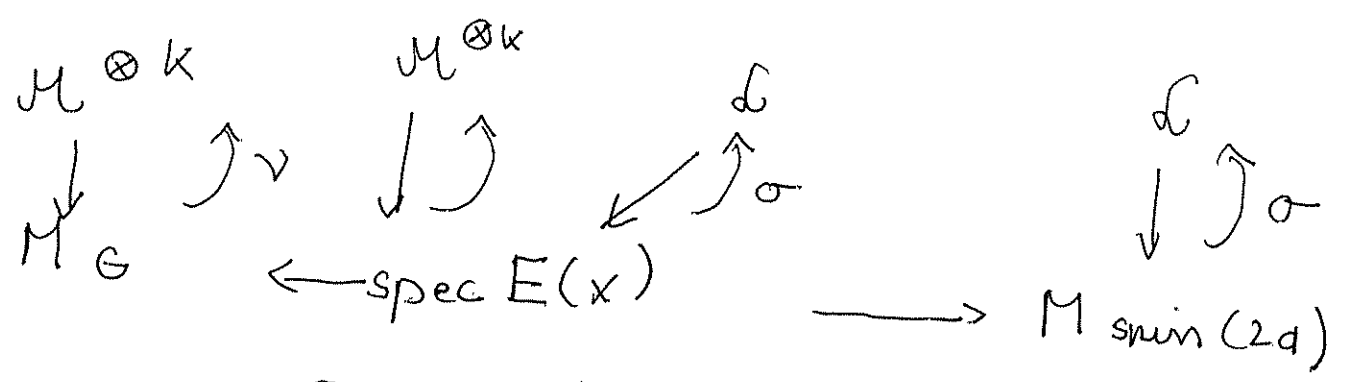
$T-V$  virtual string bdl



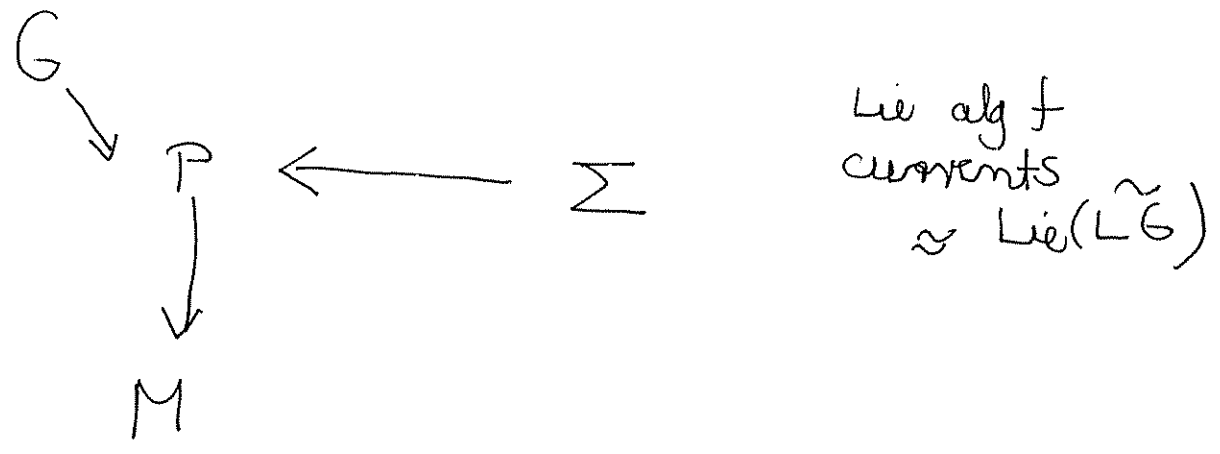
$W(M; V)$  rigid

$\iff c_2(P/M) = c_2(T)$

$$\text{spec } E(x) \longrightarrow M \begin{matrix} \mathcal{L} \\ \downarrow \\ \text{spin}(2d) \end{matrix}$$



$$\Theta \otimes \sigma^{-1} \in \Gamma(\mathcal{L} \otimes \mathcal{M}^{\otimes -k}) \cong \Gamma(\alpha_c)$$



$$c_2(\Pi) = k c_2(P)$$

$\Sigma \xrightarrow{f} G$

$\mathcal{A} \in \Omega^3$

$\int_B f^* \mathcal{A}$  (closed 3-form)

$n|_B$   
 $w/\partial B = \Sigma$

So Hilbert space of states is a Rep of  $LG$