TYPE IIB solutions with 16 Supersymmetries

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- Exact half-BPS Type IIB interface solutions I,
  Local solutions and supersymmetric Janus, arXiv:0705.0022
- Exact half-BPS Type IIB interface solutions II,
  Flux solutions and multi-Janus, arXiv:0705.0024
- Gravity duals of half-BPS Wilson loops, arXiv:0705.1004

Supersymmetry as a problem in geometry

- Supersymmetry = existence of "covariantly constant spinors"
  - Calabi-Yau manifolds C_6: at least 4 susys on $M_4 \times C_6$
  - Sasaki-Einstein manifolds SE_5: at least 4 susys on $AdS_5 \times SE_5$
- Here : Type IIB 10-dim solution manifolds preserving 16 supersymmetries
  - with a generalized connection
  - on certain classes of manifolds, ($\times_w$ is a warped product over Σ)

$$AdS_p \times S^{q_1} \times S^{q_2} \times_w \Sigma$$

  - we solve exactly two cases with dim $\Sigma = 2$
  - but a more general set of problems is probably solvable as well

The AdS/CFT Correspondence

The AdS/CFT correspondence is a conjectured equivalence between,

$\mathcal{N} = 4$ Super Yang-Mills $\Leftrightarrow$ Type IIB string theory
on flat Minkowski $\mathbf{R}^4$ on $AdS_5 \times S^5$

Yang-Mills Theory $\Leftrightarrow$ A Theory of gravity

Maldacena (1997);
Gubser, Klebanov, Polyakov (1998);
Witten (1998)

$\mathcal{N}=4$ Super Yang-Mills

- Yang-Mills theory with gauge group $SU(N)$ on flat Minkowski $\mathbf{R}^4$;

A		$SU(N)$ connection	$\mathbf{1}$ of $SU(4)_R$
ψ^a	$a=1,\cdots,4$	4 Weyl gauginos	$\mathbf{4}$ of $SU(4)_R$
ϕ^i	$i=1,\cdots,6$	6 real scalars	$\mathbf{6}$ of $SU(4)_R$

- The theory is invariant under extended Poincaré supersymmetry,

$$\{Q^a_\alpha, \bar Q^b_\beta\} = \delta^{ab}\gamma^\mu_{\alpha\beta}P_\mu \qquad a,b=1,\cdots,\mathcal{N}=4$$

which maps the fields as follows, $A \xrightarrow{Q} \psi \xrightarrow{Q} \phi \xrightarrow{Q} \psi \xrightarrow{Q} A$

- $\mathcal{N}=4$ is the maximal Poincaré supersymmetry ($=8$ susys) in YM theory.
  - $SU(4)_R$ is the automorphism group of the superalgebra.
  - Invariant under conformal transformations $SO(2,4)\sim SU(2,2)$,
  - this adds conformal supersymmetries S^a_α and $\bar S^a_\alpha$
- Full invariance group is $SU(2,2|4)\supset SO(2,4)\times SU(4)_R$

Type IIB String Theory

- Maximal number of 32 supersymmetries;
  - Massless forms of rank 0,2,4, which couple to $D-1, D1,$ $D3$ branes

- The massive string states are heavy $(\alpha')^{-\frac{1}{2}} \sim 10^{19}\times$ proton mass.
  - At energy scales $\ll (\alpha')^{-\frac{1}{2}}$, only the massless states matter;

- At low energy, Type IIB string theory reduces to Type IIB supergravity.
  - Advantage : supergravity may be described by local fields.

Type IIB Supergravity

g_{MN}	metric	$M,N=0,1,\cdots,9$	
B	axion/dilaton	contains χ,ϕ	
$B_{(2)}$	antisymm	$G_{(3)}\sim dB_{(2)}-BdB^*_{(2)}$	
$C_{(4)}$	antisymm	$F_{(5)}\sim dC_{(4)}+\mathrm{Im}(\bar B_{(2)}dB_{(2)})$	$F_{(5)}=*F_{(5)}$
ψ_M/λ	gravitino/dilatino	Weyl spinors	

- Susy variation eqs for the spinors ($\Gamma,\mathcal{B}$ generators of Cliff(1,9))

$$\delta\lambda = i(dB)\cdot\Gamma\,\mathcal{B}^{-1}\varepsilon^* - \frac{i}{4}(G_{(3)}\cdot\Gamma)\varepsilon$$

$$\delta\psi_M = D_M\varepsilon + \frac{i}{4}(F_{(5)}\cdot\Gamma)\,\Gamma_M\varepsilon - \frac{1}{16}\Big(\Gamma_M(G_{(3)}\cdot\Gamma)+2(G_{(3)}\cdot\Gamma)\Gamma_M\Big)\mathcal{B}^{-1}\varepsilon^*$$

AdS/CFT – mapping

SYM in d=4	$\Leftrightarrow$	Type IIB strings	on $AdS_5 \times S^5$
$SO(2,4)$ conformal	$=$	$SO(2,4)$	isometry of AdS_5
$SU(4)_R$ automorph.	$=$	$SU(4) \sim SO(6)$	isometry of S^5
$SU(2,2\|4)$ superconf.	$=$	$SU(2,2\|4)$	"isometry"

states $\sim$ operators	$\Leftrightarrow$	supergravity solutions
(susy inv)	$\Leftrightarrow$	(susy inv solutions)

- Solving the BPS equations $\delta\lambda = \delta\psi_M = 0$ on $AdS_5 \times S^5$
  - maximal number of 32 solutions ε
  - accounts for the 32 odd Grassmann generators of $SU(2,2|4)$

Generalizing the AdS/CFT correspondence

- Still conformal $SO(2,4)$ invariant, $AdS_5 \times SE_5$,
  - At least 4 susy requires SE_5 Sasaki-Einstein
- Only Poincaré invariance and asymptotically AdS_5
  - Physically the most interesting
  - Even with supersymmetry, this is hard – no exact solutions known
- We obtain general exact solution for geometries with 16 supersymmetries
  - $AdS_4 \times S^2 \times S^2 \times_w \Sigma$ with symmetry $SO(2,3) \times SO(3) \times SO(3)$
    CFT side : $\mathcal{N} = 4$ Yang-Mills with susy planar interface
  - $AdS_2 \times S^4 \times S^2 \times_w \Sigma$ with symmetry $SO(2,1) \times SO(5) \times SO(3)$
    CFT side : $\mathcal{N} = 4$ Yang-Mills with susy Wilson loop

Geometries with 16 susys and CFT duals

- Can one construct all solutions with 16 susys to Type IIB sugra ?
- Can one construct all solutions with 16 susys which have a CFT dual ?
  - View as AdS duals to deformations of $\mathcal{N} = 4$ SYM
  - Expect a subgroup H of $SU(2,2|4)$ with 16 susys to be preserved
  - Semi-simple H, with bosonic subgroup H_B

H	H_B	space-time	sol's
$SU(2\|2)\times SU(2\|2)$	$SO(4)\times SO(4)\times R$	$M_4\times S^3\times S^3$	LLM
$OSp(4\|4^*)$	$SO(2,3)\times SO(3)\times SO(3)$	$AdS_4\times S^2\times S^2\times\Sigma$	DEG
		$AdS_4\times S^3\times\Sigma$	?
$OSp(4^*\|4)$	$SO(2,1)\times SO(3)\times SO(5)$	$AdS_2\times S^2\times S^4\times\Sigma$	DEG
$SU(2\|4)$	$SO(3)\times SO(5)$	$M_3\times S^2\times S^4$	?
$SU(1,1\|4)$	$SO(2,1)\times SO(5)$	$AdS_2\times S^5\times E_3$	?
$SU(2,2\|2)$	$SO(2,4)\times SO(3)$	$AdS_5\times S^2\times E_3$	?

AdS dual to Interface with 16 susys

- Symmetry $SO(2,3) \times SO(3) \times SO(3)$
- Space-time is $AdS_4 \times S_1^2 \times S_2^2$ warped over a 2-dim parameter space Σ

$$ds^2 = f_1^2 ds^2_{S_1^2} + f_2^2 ds^2_{S_2^2} + f_4^2 ds^2_{AdS_4^2} + ds^2_\Sigma$$

$$G_{(3)} = \mathcal{G} \wedge f_1^2 V_{S_1^2} + i\mathcal{H} \wedge f_2^2 V_{S_2^2}$$

$$F_{(5)} = -\mathcal{F} \wedge f_4^4 V_{AdS_4} + *_\Sigma \mathcal{F} \wedge f_1^2 f_2^2 V_{S_1^2} \wedge V_{S_2^2}$$

– $ds^2_{S^2_{1,2}}$ and $ds^2_{AdS_4^2}$ unit radius metrics, $V_{S^2_{1,2}}$ and V_{AdS_4} volume forms

– f_1, f_2, f_4 are real functions, $\mathcal{F}, \mathcal{G}, \mathcal{H}$ are 1-forms on Σ
– choose local complex coordinates $w, \bar{w}$ on Σ with $ds^2_\Sigma = \rho^2 |dw|^2$

Solving via a new integrable system

- Reduce BPS eqs $\delta\lambda = \delta\psi = 0$ to the above Ansatz
  - Reduced eqs are equivalent to an integrable system on Σ,

$$\partial_{\bar w}\Big(\partial_w \vartheta - 2(\cos\mu)^{-1}(\partial_w \mu)e^{-i\vartheta}\Big) + \text{c.c.} = 0$$

  - May be exactly integrated in terms of 2 harmonic fcts h_1, h_2 on Σ,
- We obtain the general local solution in terms of h_1, h_2, e.g.

$$e^{4\phi} = \frac{2h_1h_2|\partial_w h_2|^2 - h_2^2 W}{2h_1h_2|\partial_w h_1|^2 - h_1^2 W} \qquad W \equiv \partial_w h_1 \partial_{\bar w} h_2 + \text{c.c.}$$

$$\rho^8 = \frac{W^2}{h_1^4 h_2^4}\Big(2h_1h_2|\partial_w h_2|^2 - h_2^2 W\Big)\Big(2h_1h_2|\partial_w h_1|^2 - h_1^2 W\Big)$$

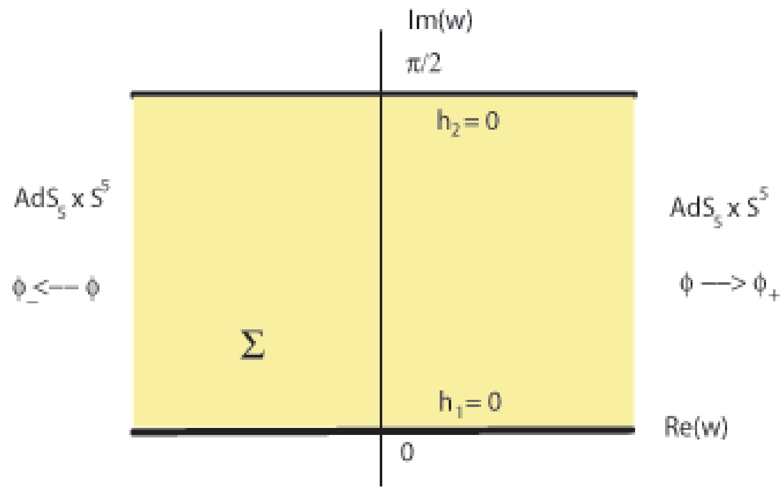
- Type IIB supergravity field equations automatically solved, locally.

$AdS_5 \times S^5$ and an interface generalization

- These regularity conditions inside Σ are obeyed by $AdS_5 \times S^5$
  - and an immediate generalization thereof
  - We readily obtain a 2-parameter family of regular solutions,

$$h_1 = \mathrm{Im}\left(e^{w-\phi_+} - e^{-w-\phi_-}\right) \qquad h_2 = \mathrm{Re}\left(e^{w+\phi_+} + e^{-w+\phi_-}\right)$$

- For $\phi_+ = \phi_-$ gives $AdS_5 \times S^5$
- For $\phi_+ \neq \phi_-$, dilaton varies
  $=$ interface solution with 16 susys
  - inside Σ : $W \leq 0$, $h_1, h_2 \geq 0$
  - on $\partial\Sigma$: $h_1 h_2 = 0$



Regularity conditions

- Adopt regularity conditions on $f_1, f_2, f_4, \rho, \phi$ and on the forms $\mathcal{F}, \mathcal{G}, \mathcal{H}$
  (R1) non-singular inside Σ
  (R2) non-singular on $\partial\Sigma$, except possibly at isolated points

- Regularity inside Σ requires that

$$0 < e^{4\phi} = \frac{2h_1 h_2 |\partial_w h_2|^2 - h_2^2 W}{2h_1 h_2 |\partial_w h_1|^2 - h_1^2 W} \qquad W \equiv \partial_w h_1 \partial_{\bar w} h_2 + \mathrm{c.c.}$$

$$0 \leq \rho^8 = \frac{W^2}{h_1^4 h_2^4} \left(2h_1 h_2 |\partial_w h_2|^2 - h_2^2 W\right) \left(2h_1 h_2 |\partial_w h_1|^2 - h_1^2 W\right)$$

– How to satisfy the regularity conditions most generally ??
– Set of manifestly sufficient conditions inside Σ,

$$h_1 > 0 \qquad h_2 > 0 \qquad W \leq 0$$

Regularity conditions cont'd

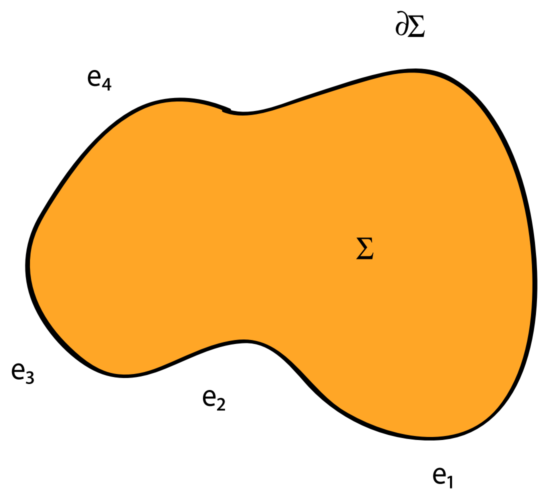
- Still need regularity conditions on the boundary $\partial\Sigma$.

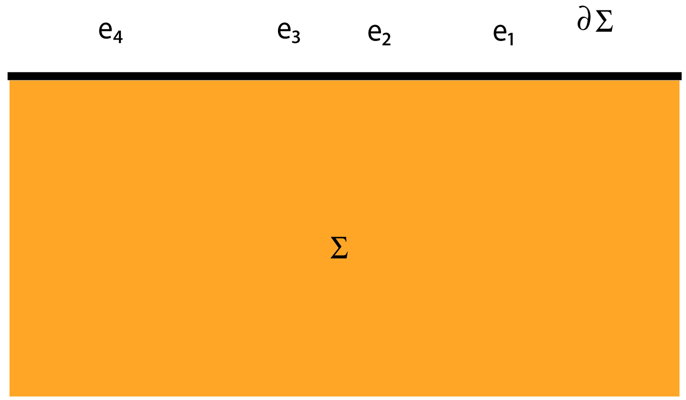
$\star$ $AdS_5 \times S^5$ regions have isolated singularities for ρ, f_4 on $\partial\Sigma$,
– but correspond to regular 10-dimensional geometry

$\star$ The probe limit of certain D-branes may be singular on $\partial\Sigma$
– We want to retain such possible solutions

- Additional assumptions :
  - ONLY singularities e_i on $\partial\Sigma$ correspond to $AdS_5 \times S^5$ regions
  - These isolated points divide the boundary into segments
  - The points on each segment must be interior points in 10-dim
  - Assume $\partial\Sigma$ connected, i.e. just a single boundary

Regularity conditions cont'd





- Map single boundary to real axis
- Segments $]e_{i+1}, e_i[$ correspond to interior points of 10 dim solution
  $\Rightarrow$ Either S_1^2 or S_2^2 must shrink to zero on $\partial\Sigma$ (but never AdS_4)
  $\Rightarrow$ Either $f_1 = 0$ or $f_2 = 0$ on $\partial\Sigma$ (but f_4 is never zero);

Linearizing the regularity conditions

- The form of the solution imposes boundary conditions on h_1, h_2,

$$
\begin{aligned}
4W^4 &= \rho^4 f_1^2 f_2^2 & \qquad W &= 0 \quad \text{on} \quad \partial\Sigma \\
f_1^2 f_4^2 &= 4e^{+2\phi}\, h_1^2 & \qquad f_1 = 0 \quad &\Rightarrow \quad (h_1 = 0 \;\;\&\;\; \partial_n h_2 = 0) \\
f_2^2 f_4^2 &= 4e^{-2\phi}\, h_2^2 & \qquad f_2 = 0 \quad &\Rightarrow \quad (h_2 = 0 \;\;\&\;\; \partial_n h_1 = 0)
\end{aligned}
$$

- Equivalent to two coupled electro-statics problems with
  - alternating Neumann and vanishing Dirichlet conditions on $\partial\Sigma$
  - $\partial_w h_1, \partial_w h_2$ alternating real or imaginary on $\partial\Sigma = \mathbf{R}$
  - $h_1, h_2 > 0$ and $W \leq 0$ inside Σ

Solving regularity conditions by hyperelliptic surfaces

- Map the domain Σ onto the lower half-plane with complex coordinate u.
  - The boundary $\partial\Sigma$ is then the real axis $\mathbf{R}$.
  - Points e_i on $\partial\Sigma$ where Dirichlet $\leftrightarrow$ Neumann, $i = 1, 2, \cdots, 2g+2$.
- Construction of $\partial h_1, \partial h_2$ via hyperelliptic curve of genus g, defined by

$$s(u)^2 = (u - e_1)(u - e_2) \cdots (u - e_{2g+1})$$

$$e_{2g+1} < \cdots < e_1 < e_0 = \infty$$

  - $s(u)^2$ changes sign across each branch point e_i
  - $s(u)$ alternates between real and imaginary on $\partial\Sigma = \mathbf{R}$
- The holó differential $du/s(u)$ alternates between real and imaginary
  - but it does not have the proper asymptotics at e_i

Solving regularity conditions cont'd

- The meromorphic differentials $\partial h_1, \partial h_2$ may be written down explicitly,

$$\partial h_1 = -i\frac{P_1(u)du}{s(u)^3} \qquad \partial h_2 = -\frac{P_2(u)du}{s(u)^3}$$

  - for two real polynomials P_1, P_2,
  - Neumann and Dirichlet conditions satisfied by construction,
  - $W = 0$ on $\partial\Sigma$
  - behavior at branch points $du/(u-e_i)^{3/2}$ guarantees asymptotic $AdS_5 \times S^5$
  - behavior at branch point $e_0 = \infty$ requires degrees of P_1, P_2 equal $3g+1$

- It remains to enforce
  - $W \leq 0$ inside Σ
  - $h_1 > 0$, $h_2 > 0$ inside Σ
  - the vanishing of the Dirichlet boundary conditions

Solving regularity conditions cont'd

- $W \leq 0$ inside Σ
  - All complex zeros of $P_1(u), P_2(u)$ must be common
  - Otherwise $W \leq 0$ cannot maintain constant sign near a zero
  - $P(u) = \prod_{a=1}^{p}(u - u_a)(u - \bar{u}_a)$ with $\mathrm{Im}(u_a) < 0$

$$\partial h_1 = -i\frac{P(u)Q_1(u)du}{s(u)^3} \qquad \partial h_2 = -\frac{P(u)Q_2(u)du}{s(u)^3}$$

  - $Q_1(u)$ has only real zeros $\alpha_q < \cdots < \alpha_2 < \alpha_1$ $\qquad 2p + q = 3g + 1$
  - $Q_2(u)$ has only real zeros $\beta_q < \cdots < \beta_2 < \beta_1$

  - condition $W \leq 0$ reduces to $\mathrm{Im}(Q_1(u)Q_2(\bar{u})) > 0$
  - solved uniquely by ordering α, β,

$$\alpha_q < \beta_q < \alpha_{q-1} < \beta_{q-1} < \cdots < \alpha_2 < \beta_2 < \alpha_1 < \beta_1$$

Solving regularity conditions cont'd

- $h_1 > 0, \; h_2 > 0$ near the branch points e_i requires
  - $Q_1(e_{4j}), Q_1(e_{4j+1}), Q_2(e_{4j}), Q_2(e_{4j+3}) > 0$
  - $Q_1(e_{4j+2}), Q_1(e_{4j+3}), Q_2(e_{4j+1}), Q_2(e_{4j+2}) < 0$
- $h_1 > 0, \; h_2 > 0$ near entire $\partial\Sigma$ gives further conditions
  - uniquely solved by relative ordering of e, α, β

$$\alpha_{g+1} < e_{2g+1} < \beta_{g+1} < e_{2g} < \cdots < e_2 < \alpha_1 < e_1 < \beta_1$$

- It only remains to ensure that the Dirichlet conditions VANISH,

$$\mathrm{Re} \int_{e_{2j}}^{e_{2j-1}} \partial h_1 = \mathrm{Re} \int_{e_{2j+1}}^{e_{2j}} \partial h_2 = 0 \qquad\qquad j = 1, \cdots, g$$

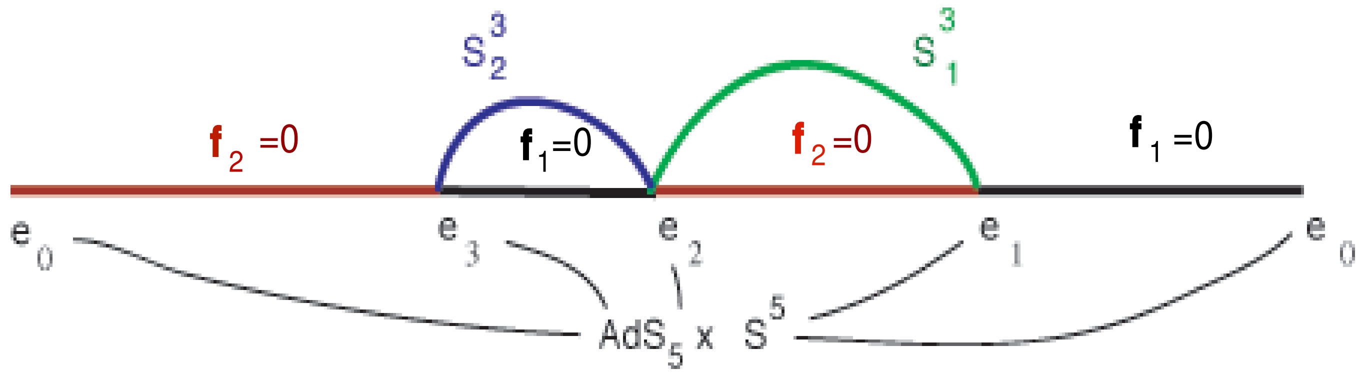
- Given the branch points e_i and the ordered real zeros α_b, β_b,
  - Period relations determine g complex zeros u_a IF a solution exists

  $\Rightarrow$ mathematical problem of determining the moduli space of solutions
  - The geometry of the allowed moduli space is known explicitly for $g = 1$

Topology of regular solutions

- $2g+2$ branch points = different asymptotic boundary $AdS_5 \times S^5$ regions
  - each with its independent constant dilaton limit
  - Number of free parameters of solution is $4g+6$
- There are g independent pairs of homology 3-spheres, $j = 1, \cdots, g$
  - $S^3_{1j} = [e_{2j}, e_{2j-1}] \times_f S^2_1$ NSNS 3-form charges $\int_{S^3_{1j}} H_{(3)}$
  - $S^3_{2j} = [e_{2j+1}, e_{2j}] \times_f S^2_2$ RR 3-form charges $\int_{S^3_{2j}} F_{(3)}$
- The presence of 3-form fluxes reveals underlying D5 and NS5 branes
  - These solutions are fully back-reacted D5 and NS5 branes
  - in the presence of D3 branes in the near-horizon limit

– e.g. genus 1



The Genus 1 Solution

- On the lower half-plane, u,

$$Q_1(u) = (u-\alpha_1)(u-\alpha_2) \qquad s(u)^2 = (u-e_1)(u-e_2)(u-e_3)$$

$$Q_2(u) = (u-\beta_1)(u-\beta_2) \qquad P(u) = (u-u_1)(u-\bar u_2)$$

  – with the ordering $\alpha_2 < e_3 < \beta_2 < e_2 < \alpha_1 < e_1 < \beta_1 < e_0 = \infty$
- View the vanishing Dirichlet conditions as eqs for unknown u_1,

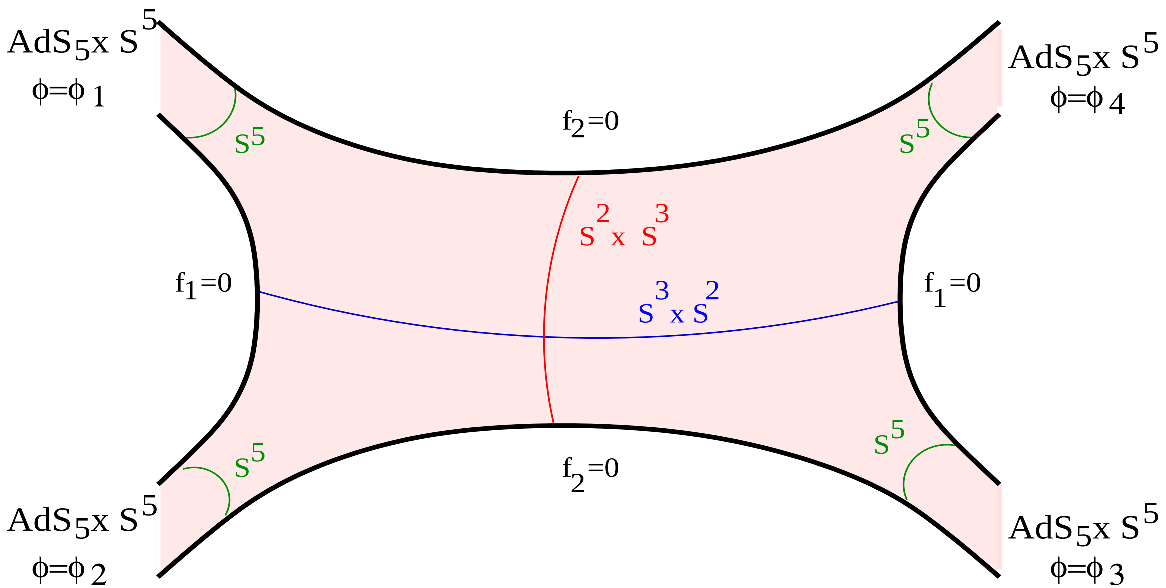
$$\begin{aligned} a_0|u_1|^2 - a_1(u_1+\bar u_1) + a_2 &= 0 \\ b_0|u_1|^2 - b_1(u_1+\bar u_1) + b_2 &= 0 \end{aligned}$$

- The $a_0, a_1, a_2, b_0, b_1, b_2$ are modular connections,

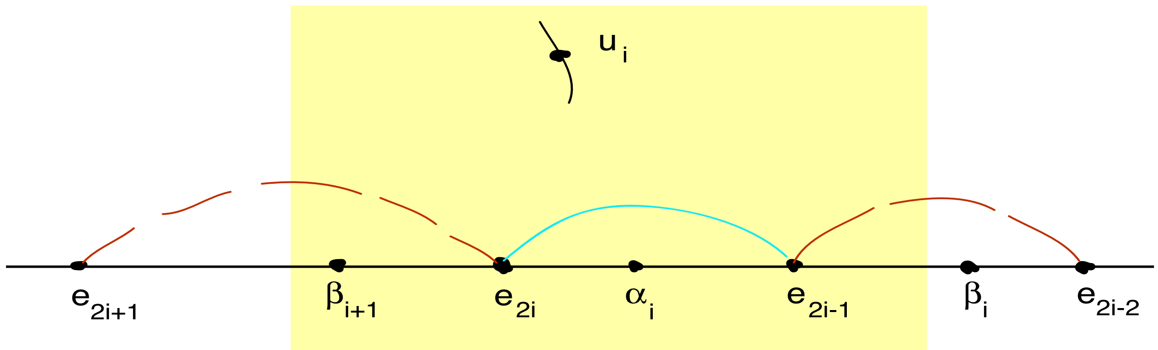
$$a_n = \delta_{n,1} + (\alpha_1+\alpha_2+\zeta_3)\delta_{n,2} + \sum_{i=1}^{3} e_i^n (e_i+\zeta_3) Q_1(e_i)/E_i^2$$

  – with $\zeta_i = \zeta(\omega_i)/\omega_i$, $E_i = (e_i-e_j)(e_i-e_k)$ and (ijk) is a perm. of (123)

Genus 1 solution cont'd



Topology change : a collapsing branch cut



$$\partial h_1 = \frac{(u-u_i)(u-\bar{u}_i)(u-\alpha_i)}{(u-e_{2i})^{3/2}(u-e_{2i-1})^{3/2}} (\partial h_1)_{g-1} \qquad \partial h_2 = \frac{(u-u_i)(u-\bar{u}_i)(u-\beta_{i+1})}{(u-e_{2i})^{3/2}(u-e_{2i-1})^{3/2}} (\partial h_2)_{g-1}$$

- As $e_{2i-1} \to e_{2i}$ we must have $\alpha_i \to e_{2i}$, and $\mathrm{Im} \int \partial h_1 = 0$ forces $u_i \to e_{2i}$
- Two possibilities
  (A) $\beta_{i+1} \to e_{2i}$ gives topology change $(\partial h_{1,2})_g \to (\partial h_{1,2})_{g-1}$
  (B) $\beta_{i+1} \not\to e_{2i}$ gives $\partial h_1 \to (\partial h_1)_{g-1}$ but leaves a singular ∂h_2
  $\sim$ the probe limit: a D5 (or NS5) brane remains

Total branch cut collapse

- Collapse of all branch cuts produces limit with singular branes,
  - m_R D5 branes and m_{NS} NS5 branes with $m_R + m_{NS} = g$
  - leads to a simple explicit solution, for all genera g,

$$h_1 = -2i(w - \bar{w}) \left(1 + \frac{C_0}{|w|^2}\right) + \sum_{j=1}^{m_R} \frac{C_j}{\ell_j} \ln \left|\frac{w + i\ell_j}{w - i\ell_j}\right|^2$$

$$h_2 = -2(w + \bar{w}) \left(1 + \frac{D_0}{|w|^2}\right) - \sum_{i=1}^{m_{NS}} \frac{D_i}{k_i} \ln \left|\frac{w + k_i}{w - k_i}\right|^2$$

  - for arbitrary real positive $k_i, \ell_j, C_j, D_i, C_0, D_0$,
  - RR and NSNS 3-form charges given by $\mathcal{F}_j = C_j/\ell_j$ and $\mathcal{H}_i = D_i/k_i$

Physicist's proof of existence of solutions for all genera:
- regularize all poles into branch cuts (local !)

$\Rightarrow$ there exists an open set of regular solutions
in the moduli space around these singular solutions

CFT dual to AdS_4 solutions (in progress)

- The AdS_4 factor indicates the presence of an interface.
- For $g = 0$, CFT dual has interface operators (built from bulk fields).
- For $g \geq 1$, several gauge groups
  - different species of $\mathcal{N}=4$, decoupled away from interface
  - interact only via the interface
  - are coupled via extra massless fields on the interface
  - On AdS side, extra massless fields arise from S^3 shrinking to zero
- For $g \geq 1$, as branch cuts collapse,
  - and we approach the limit with probe branes,
  - recover massless string excitations from probe D5/NS5 branes
    of De Wolfe, Freedman, Ooguri – Skenderis, Taylor
- Our solutions are fully back-reacted geometries with D5 and NS5 branes

Open Mathematical Problems

- Half-BPS solutions to Type IIB supergravity are surprisingly manageable;
- How to describe the moduli space of genus $g > 1$ solutions ?
- Regular solutions with different 10-dim topologies ?
- Unified approach to 16 susy solutions from subgroups of $SU(2,2|4)$?