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Normal Functions & Disk Counting

Based on J. Walcher hep-th/0605162
DRM & J. Walcher

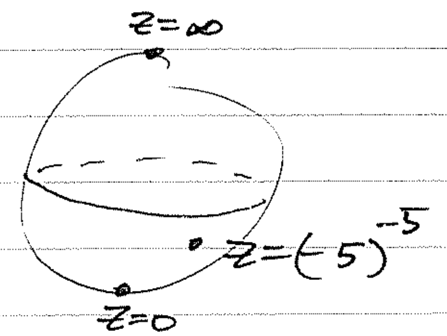
Quintic & Mirror

- quintic: $X = \{F_5(x_1,\dots,x_5)=0\} \subset \mathbb{CP}^4$ CY 3-fold
- "complexified Kähler moduli space:" $t \in H^2(X,\mathbb{C})$ w/ $\mathrm{Im}\, t$ a Kähler class
- $q = e^{2\pi i t} \in \mathcal{U} \subset H^2(X, \mathbb{C}/\mathbb{Z})$
- quintic-mirror : singular

resolved $Y \to \bar{Y} = \{\sum x_j^5 - 5\psi \prod x_j = 0\}/(\mathbb{Z}_5)^3 \subset \mathbb{CP}^4/(\mathbb{Z}_5)^3$

- ordinary ("cplx str") moduli spc has param $z = (-5\psi)^{-5}$
- Identification btwn two made w/ help of periods

$\phi(z) = \int_\Gamma \Omega_z$, $\Gamma \in H_3(Y,\mathbb{Z})$, Ω_z holo 3-form on Y_z



$\phi(z)$ satisfies algebr diff eq $D\phi = 0$, D 4^{th} order

easy to find 1 power series sol'n near $z=0$

$$\phi_0(z) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} z^n$$

other 3 sol'ns elusive

Formal power series

$$\phi(z,\alpha) = \sum_{n=0}^{\infty} (\text{mess}) z^{\alpha+n}$$

$D(\phi(z,\alpha)) = \alpha^4 z^\alpha \Rightarrow \alpha^4 = 0$ for sol'n

$\Rightarrow$ sol'n depends on $\mathbb{C}[\alpha]/(\alpha^4)$

$$\frac{(5\alpha+1)(5\alpha+2)\cdots(5\alpha+5n)}{[(\alpha+1)(\alpha+2)\cdots(\alpha+n)]^5}$$

evaluated in ring, poly of deg 3

$z^\alpha = 1 + \alpha \log z + \cdots$

$$\Rightarrow \phi(z,\alpha) = \Phi_0(z) + \alpha\phi_1(z) + \alpha^2\Phi_2(z) + \alpha^3\phi_3(z)$$

mirror map: $t = \frac{1}{2\pi i}\frac{\phi_1(z)}{\Phi_0(z)}$ i.e. $q = \exp\left(\frac{\phi_1(z)}{\Phi_0(z)}\right)$

<u>3 pt fns</u> (topological) trilinear map on $H^{1,1}(X,\mathbb{C})$, $H^1(Y, T_Y^{(1,0)})$

on X: $A, B, C \in H^{1,1}(X,\mathbb{C})$

$$\langle \mathcal{O}_A \mathcal{O}_B \mathcal{O}_C \rangle = A\cdot B\cdot C + \sum_{0\neq\eta\in H_2(X,\mathbb{Z})} A(\eta)B(\eta)C(\eta) N_\eta \frac{z^\eta}{1-z^\eta}$$

up to some integrals, really $\int_X A\wedge B\wedge C$

counts # genus 0 holo curves in η, related to Gromov-Witten inv't of X

on mirror Y: $\alpha, \beta, \gamma \in H^1(Y, T_Y^{(1,0)})$

$$\langle \mathcal{O}_\alpha \mathcal{O}_\beta \mathcal{O}_\gamma \rangle = \int_Y \nabla_\alpha (\Omega \lrcorner \beta) \wedge (\Omega \lrcorner \gamma)$$ calc. from periods

$\rightsquigarrow$ predictions of Candelas, de la Ossa, Green, Parkes

2875 lines
609250 conics
317206375 twisted cubics

Disk Counting so far, just closed strings, but w/ D-branes, open strings also important

D-branes: $L \subset X$ special Lagrangian submflds

insted of counting holo curves of fixed genus, should count open Riem. surf ending on L

$X = \{x_1^5 + x_2^5 + \dots + x_5^5 = 0\} \subset \mathbb{CP}^4$
$L = \{x_1^5 + x_2^5 + \dots + x_5^5 = 0\} \subset \mathbb{RP}^4$
L topology of $\mathbb{RP}^3$

this component contains Fermat quintic

some holo $g=0$ curves on X defined over $\mathbb{R}$
these meet L, count these
others in cplx conj. pairs

only do count for odd degree curves

actually need $U(1)$ bundle on L w/ flat connection

$H_1(L,\mathbb{Z}) = \mathbb{Z}_2 \Rightarrow 2$ choices L_+, L_-

~~ ~~ $\operatorname{im}(L_\pm)$ in $\mathrm{Fuk}(X)$ same $\rightsquigarrow$ normal fn

$$\mathcal{T}(t) = \frac{t}{2} \pm \left(\frac{1}{4} + \frac{1}{2\pi^2} \sum_{d\ \mathrm{odd}} n_d q^{d/2}\right) \quad q = e^{2\pi i t}$$

open gromov-witten invariants of degree d for disks

domain wall tension for wall separating L_+ and L_- vacua boundary conds

$$L_+ - L_- \text{ trivial in } \mathrm{Fuk}(X)$$

$\updownarrow$ mirror

? trivial in $\overset{\text{bounded derived category}}{D^b(\mathrm{Coh}\,Y)} \longrightarrow K(Y)$

$$\begin{pmatrix}\text{Branes on} \\ \text{Fermat quintic} \\ X\end{pmatrix} \overset{\text{mirror}}{\underset{\cong}{\longleftrightarrow}} \begin{pmatrix}\text{Branes on} \\ \text{Fermat mirror} \\ Y\end{pmatrix}$$

$\Updownarrow$

matrix factorizations

$$\left(\sum x_i^5 - 5\psi \prod x_i\right) I_N = \begin{pmatrix}\text{matrix w/} \\ \text{poly} \\ \text{entries}\end{pmatrix}\begin{pmatrix}\text{matrix w/} \\ \text{poly} \\ \text{entries}\end{pmatrix}$$

$\Rightarrow$ mirror of $L_+ - L_- = \mathcal{L}$ element of $D^b(\mathrm{Coh}\,Y)$

(8 term complex, mess)

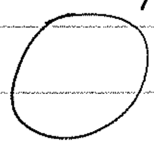
$\mathrm{ch}(\mathcal{L}) = 0$ in K-theory

$\mathrm{ch}^{\mathrm{alg}}(\mathcal{L}) \neq 0$, represented by $\underbrace{C_+ - C_-}_{\text{two holo. curves on } Y}$

$$C_+ - C_- = \partial\Gamma \quad \Gamma \text{ 3-chain}$$

$$J(z) = \int_\Gamma \Omega_z$$ well defined as element of family of int. Jacobians over $\{z\}$

$$\mathcal{H}^3 \supseteq \mathcal{H}^{3,0+2,1+1,2} \supseteq \mathcal{H}^{3,0+2,1} \supseteq \mathcal{H}^{3,0}$$



$$(H^{3,0}+H^{2,1})/H_3(\,,\mathbb{Z})$$

this only works when Fermat point

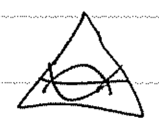
$$\partial_\alpha J(z) \quad \int_\Gamma Y_z$$

$$\overline{Y} = (x_1^5 + x_2^5 + \dots x_5^5 - 5\psi x_1 x_2 \cdots x_5)/(\mathbb{Z}_5)^3$$

$\overline{C_+}, \overline{C_-}$: $x_1 + x_2 = x_3 + x_4 = 0$ (a $\mathbb{CP}^2 \subseteq \mathbb{CP}^4$)

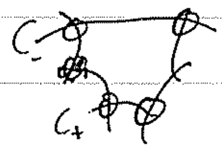
$$\begin{aligned} 0 &= x_5^5 - 5\psi x_1^2 x_3^2 x_5 \\ &= x_5 \underbrace{(x_4^2 - \sqrt{5\psi}\, x_1 x_3)}_{\overline{C_+}} \underbrace{(x_4^2 + \sqrt{5\psi}\, x_1 x_3)}_{\overline{C_-}} \end{aligned}$$

$$\overline{C_+} - \overline{C_-} = \partial \overline{\Gamma_3}$$



lift

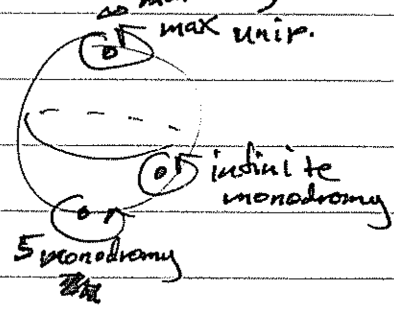
$$C_+ - C_- = \partial \Gamma_3$$



if $\Gamma_3 \subseteq \mathbb{CP}^3$

$$\Rightarrow \int_{\Gamma_3} \Omega = 0$$

$$\boxed{\underbrace{\mathcal{D}}_{\text{gets rid of periods}} \int_\Gamma \Omega_{\cancel{z}} = \frac{\overset{\#\,\text{disks}}{15}}{16\pi^2}\sqrt{z}}$$



$$\underbrace{\mathcal{D}'}_{\text{i.e. } 5^{th} \text{ order}} \int_\Gamma \Omega_z = 0$$

$$\mathcal{D}\Phi = 0 \Rightarrow \mathcal{D}'\Phi = 0 \quad \rightsquigarrow \quad \mathcal{J}(t) = \ldots$$