Modular Equations and Special Function Transformations

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High-Level Overview

- Picard–Fuchs equations:
  - Let $\mathcal{V} \xrightarrow{\pi} X \ni x$ be a family of n-dimensional algebraic varieties, with smooth fibres over the complement of a finite set.
  - Choose a holomorphic n-form ω on a smooth fibre $V_0 \in \mathcal{V}$, and n-cycles $\gamma_1, \ldots, \gamma_r$ that give a basis for its nth homology.
  - Then ω can be extended to a meromorphic family of n-forms $\omega(x)$, and the cycles (homology classes) to (multivalued!) functions of x.
  - The *periods* $\int_{\gamma_i(x)} \omega(x)$ are multivalued too, but satisfy a *Fuchsian ODE* on X (the P–F equation). They are special functions.

- When $\mathcal{V} \to X$ is a family of elliptic curves, e.g., $\mathcal{E}_N \to X_0(N)$, for $X_0(N) \cong \mathbb{P}^1(\mathbb{C})$, then covering and modular relations, e.g., the coverings $X_0(MN)/X(N)$, induce relations among P–F equations and their solutions. That is, they yield special function identities.

Based On...

- Recent work of RM, e.g.,
  - “On Rationally Parametrized Modular Equations,” arXiv:math/0611041.
  - “Algebraic Hypergeometric Transformations of Modular Origin,” *Trans. AMS* 359 (2007), 3859–3885.
  - “The 192 Solutions of the Heun Equation,” *Math. of Computation* 76 (2007), 811–843.

- See also:
  - Papers on ODEs and PDEs satisfied by automorphic forms on modular subgroups, by H. Verrill.
  - Modular parametrizations of lattice-polarized K3 surfaces, by C. Doran et al.

One-Argument Special Functions

- Functions in the function field of an algebraic curve $X/\mathbb{C}$.
(E.g., meromorphic functions on $\mathbb{P}^1(\mathbb{C})$. Or on an elliptic curve $E/\mathbb{C}$, meromorphic functions such as Jacobi's sn or Weierstraß's $\wp$.

- Functions satisfying linear homogeneous ODEs on $X/\mathbb{C}$, with meromorphic coefficients.
  - ◇ Scalar equations, e.g., $\left[\sum_{j=0}^{N} A_j(x) {D_x}^j\right] y = 0$, and
  - ◇ Systems of 1st-order equations, e.g., $D_x y^{(i)} - \sum_{j=1}^{N} {A^i}_j(x) y^{(j)} = 0$, intepretable in terms of a connection on a rank-N vector bundle over $X/\mathbb{C}$. Their solutions come 'from geometry.'

- In particular, the case when $X/\mathbb{C}$ is the base of a family of algebraic varieties $\mathcal{V} \xrightarrow{\pi} X$. (E.g., a Picard–Fuchs equation.)

The GHE and HE

- The GHE (Gauss hypergeometric equation) is the canonical linear 2nd-order ODE on $\mathbb{P}^1(\mathbb{C})$ with three regular singular points, and the GHE (Heun equation) is the one with four.
  - ◇ The singular points are $x = 0, 1, \infty$ by convention; and (for the Heun equation) $x = a$, for some $a \in \mathbb{C} \setminus \{0, 1\}$.
  - ◇ Characteristic exponents (whence monodromy) are canonicalized.
  - ◇ The HE has an extra degree of freedom: an accessory parameter.

- The standard solutions of the GHE and HE (analytic at $x = 0$, normalized to unity there) are ${}_2F_1$ and Hl.
  - ◇ Their Taylor coefficients at $x = 0$ satisfy 2-term and 3-term recurrences, respectively.

The GHE $\mathcal{E}(a,b;c)$ and Its Series Solution

$$D_x^2\,u + \left[\frac{c}{x} + \frac{a+b-c+1}{x-1}\right] D_x u + \left[\frac{ab}{x(x-1)}\right] u = 0.$$

The characteristic exponents at $x = 0, 1, \infty$ are:
$0, 1-c$; $0, c-a-b$; a, b. Each has an associated Frobenius solution.

The zero-exponent solution at $x = 0$, normalized, is

$${}_2F_1(a,b;\,c;x) := \sum_{n=0}^{\infty} c_n x^n,$$

converging on $|x| < 1$, where $c_0 = 1$ and

$$(n+a)(n+b)\,c_n - (n+1)(n+c)\,c_{n+1} = 0.$$

The HE and Its Series Solution

$$D_x^2 u + \left[\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a}\right] D_x u + \left[\frac{\alpha\beta\, x - q}{x(x-1)(x-a)}\right] u = 0.$$

The characteristic exponents at $x = 0, 1, a, \infty$ are:
$0, 1-\gamma$; $0, 1-\delta$; $0, 1-\epsilon$; α, β. Each has an associated Frobenius solution.

- ◇ By Fuchs's relation, $\alpha + \beta - \gamma - \delta + \epsilon + 1 = 0$.
- ◇ $q \in \mathbb{C}$ is an *accessory* (non-exponent-related) parameter.

The zero-exponent solution at $x = 0$, normalized, is

$$Hl(a, q;\, \alpha, \beta, \gamma, \delta;\, x) := \sum_{n=0}^{\infty} c_n x^n.$$

Heun Series

$$Hl(a, q;\, \alpha, \beta, \gamma, \delta;\, x) := \sum_{n=0}^{\infty} c_n x^n,$$

converging on $|x| < \min(1, |a|)$, where $c_0 = 1$, and (with $c_{-1} := 0$)

$$\begin{aligned}(n+\alpha-1)(n+\beta-1)\, c_{n-1} \\ - \Big\{ n \left[\, (n+\gamma+\delta-1)a + (n+\gamma+\epsilon-1) \,\right] + q \Big\}\, c_n \\ + (n+1)(n+\gamma)a\, c_{n+1} = 0.\end{aligned}$$

Claim: Any generic series $\sum_{n=0}^{\infty} c_n x^n$ in which $\{c_n\}_{n=0}^{\infty}$ satisfy a 3-term recurrence relation, with coefficients quadratic in n, is of Heun type.

Some History

- Heun (1889) first wrote down and studied the HE.
  - ◇ The Lamé equation is a special case of it.
    - $*$ See RM, *Philos. Trans. Roy. Soc. A* 366 (2008), 1115–1153.
  - ◇ Confluent HEs have also been studied (Slavyanov et al.).
- A long-term goal: deriving, for Hl, analogues of ${}_2F_1$ identities. E.g.,
  - ◇ Degree-1 rational transformations of Hl, arising from Möbius automorphisms of $\mathbb{P}^1(\mathbb{C})$. (Cf. Kummer's 24 solutions of the GHE.)
  - ◇ Higher-degree rational transformations (quadratic, etc.) of Hl. (Cf. Kummer's quadratic transformations of ${}_2F_1$, Goursat's, etc.)
  - ◇ Algebraic transformations of Hl. (Not classified even for ${}_2F_1$!).
  - ◇ Contiguity relations ("Schlesinger transformations"), etc.

Degree-1 Rational Transformations

Kummer's 24 Series Solutions of the GHE

Each of the 6 Frobenius solutions of the GHE can be written in four equivalent ways, in terms of ${}_2F_1$.

- Example: the zero-exponent solution at $x = 0$ can be written as

$$ {}_2F_1(a, b; c;\ x), \qquad (1-x)^{-a-b+c}\, {}_2F_1(c-a, c-b; c;\ x), $$
$$ (1-x)^{-a}\, {}_2F_1(a, c-b; c;\ \tfrac{x}{x-1}), \qquad (1-x)^{-b}\, {}_2F_1(c-a, b; c;\ \tfrac{x}{x-1}). $$

Cf. *Euler's transformation* and *Pfaff's transformation* of ${}_2F_1$.

- Example: the zero-exponent solution at $x = 1$ can be written as

$$ {}_2F_1(a, b; a+b-c+1;\ 1-x), \qquad x^{1-c}\, {}_2F_1(b-c+1, a-c+1; a+b-c+1;\ 1-x), $$
$$ x^{-a}\, {}_2F_1(a, a-c+1; a+b-c+1;\ \tfrac{x-1}{x}), \qquad x^{-b}\, {}_2F_1(b-c+1, b; a+b-c+1;\ \tfrac{x-1}{x}). $$

The Kummer Transformations of the GHE

- The GHE $\mathcal{E}(a,b;c)$ is transformed to $\mathcal{E}(a',b';c')$ by

  ① Möbius transformations of the independent variable x that preserve the set of singular points $\{0,1,\infty\}$; i.e.,
  $x \mapsto x$, $1-x$, $x/(x-1)$, $1/(1-x)$, $x/(x-1)$, $1/x$.

  ② Changes of the dependent variable: 'index flips', i.e., *characteristic exponent negations*, such as

$$(1-x)^{-\theta_1}\left\{\begin{array}{ccc|c} 0 & 1 & \infty & x \\ \hline 0 & 0 & a & \\ \theta_0 & \theta_1 & b & \end{array}\right\} = \left\{\begin{array}{ccc|c} 0 & 1 & \infty & x \\ \hline 0 & -\theta_1 & a+\theta_1 & \\ \theta_0 & 0 & b+\theta_1 & \end{array}\right\}$$

- The 4 variants of each of the 6 Frobenius solutions, in terms of ${}_2F_1$, are transformed among by composite transformations that (i) stabilize $x=0,1,$ or ∞, and (ii) perform no F-homotopy there.

The Kummer Automorphism Group of the GHE

- The subgroup of Möbius transformations is isomorphic to S_3.

- It normalizes the subgroup of index flips, isomorphic to $(\mathbb{Z}_2)^3$.
  (Or merely to $(\mathbb{Z}_2)^2$, since the interchange of exponents at $x = \infty$, i.e., $a \leftrightarrow b$, is trivial.)

$\Longrightarrow$ The Kummer group of composite transformations is isomorphic to an order-48 group, the *wreath product* $\mathcal{B}_3 = \mathbb{Z}_2 \wr S_3 = (\mathbb{Z}_2)^3 \rtimes S_3$.
(Or merely to an index-2 subgroup $\mathcal{D}_3 = [\mathbb{Z}_2 \wr S_3]_{\mathrm{even}}$, of order 24.)

$\Longrightarrow$ The Kummer group is the group of *signed* permutations of 3 objects.
(The index-2 subgroup is the *even-signed* subgroup: $\mathcal{D}_3 \cong S_4$.)

(The 4 variants of each Frobenius solution are transformed among by a $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup of $\mathcal{D}_3$.)

The Indexing of Kummer's 24 Series Solutions

- Example: The four equivalent expressions for the zero-exponent solution at $x = 0$,

$$ {}_2F_1(a,b;c;\,x), \qquad (1-x)^{-a-b+c}\,{}_2F_1(c-a,c-b;c;\,x), $$
$$ (1-x)^{-a}\,{}_2F_1(a,c-b;c;\,\tfrac{x}{x-1}), \qquad (1-x)^{-b}\,{}_2F_1(c-a,b;c;\,\tfrac{x}{x-1}). $$

are indexed by $[0_+][1_+][\infty_+]$, $[0_+][1_-][\infty_-]$, $[0_+][1_+\infty_+]$, $[0_+][1_-\infty_-]$.

- Example: The four equivalent expressions for the zero-exponent solution at $x = 1$,

$$ {}_2F_1(a,b;a+b-c+1;\,1-x), \qquad x^{1-c}\,{}_2F_1(b-c+1,a-c+1;a+b-c+1;\,1-x), $$
$$ x^{-a}\,{}_2F_1(a,a-c+1;a+b-c+1;\,\tfrac{x-1}{x}), \qquad x^{-b}\,{}_2F_1(b-c+1,b;a+b-c+1;\,\tfrac{x-1}{x}). $$

are indexed by $[1_+0_+][\infty_+]$, $[1_+0_-][\infty_-]$, $[1_+0_+\infty_+]$, $[1_+0_-\infty_-]$.

The Extension to the HE (4 Singular Points, Not 3)

- The HE $\mathcal{E}(a, q; \alpha, \beta, \gamma, \delta)$ is transformed to $\mathcal{E}(a', q'; \alpha', \beta', \gamma', \delta')$ by

  ① Möbius transformations of the independent variable x that 'preserve' the singular points, i.e., take $\{0, 1, a, \infty\}$ to $\{0, 1, a', \infty\}$.

  E.g., $x \mapsto x$, $1 - x$, etc., and x/a, $x/(x-a)$, $(1-a)x/(x-a)$, etc. These make up a subgroup isomorphic to S_4.

  ② Index flips, which are exponent negations at $x = 0, 1, a$. This subgroup is isomorphic to $(\mathbb{Z}_2)^3$. (If the $\alpha \leftrightarrow \beta$ exponent interchange at $x = \infty$ is included, the group is $(\mathbb{Z}_2)^4$.)

$\Longrightarrow$ The group of composite automorphisms is therefore isomorphic to an order-384 group, the *wreath product* $\mathcal{B}_4 = \mathbb{Z}_2 \wr S_4 = (\mathbb{Z}_2)^4 \rtimes S_4$. (Or merely to the even-signed subgroup $\mathcal{D}_4 = [\mathbb{Z}_2 \wr S_4]_{\mathrm{even}}$, of order 192, if $\alpha \leftrightarrow \beta$ is dropped.)

Algebraic Transformations

#1: Landen's Transformation $[X_0(8)/X_0(4)]$

- The (first) complete elliptic integral $\mathsf{K}_2 = \mathsf{K}_2(\alpha)$, defined by

$$\mathsf{K}_r(\alpha) \propto \int_0^1 t^{-1/r}(1-t)^{-1+1/r}(1-\alpha t)^{-1/r}\,dt$$
$$\propto \underline{{}_2F_1(1/r, 1-1/r;\, 1;\, \alpha)}$$

satisfies

$$\mathsf{K}_2(\alpha) = (2/\alpha)(1-\sqrt{1-\alpha})\ \mathsf{K}_2(\beta),$$

provided

$$\alpha^2(1-\beta)^2 - 16(1-\alpha)\beta = 0.$$

Here α, β are confined to a neighborhood of $(0,1)$ in $\mathbb{P}^1(\mathbb{C})$.

- The algebraic α–β relation can be *uniformized*:

$$\alpha = x(x+8)/(x+4)^2, \qquad \beta = x^2/(x+8)^2.$$

#2: Another Algebraic ${}_2F_1$ Transformation $[X_0(25)/X_0(5)]$

Let $f_5(z) = \sum_{n=0}^{\infty} c_n (z/500)^n$, for $|z|$ sufficiently small, where

$$500(2n-1)^2\, c_{n-1} + 2(44n^2 + 22n + 5)\, c_n + (n+1)^2\, c_{n+1} = 0,$$

with $c_{-1} = 0$, $c_0 = 1$. Then for all x in a neighborhood of 0,

$$f_5\left(x(x^4 + 5x^3 + 15x^2 + 25x + 25)\right)$$
$$= 5\left[x^4 + 5x^3 + 15x^2 + 25x + 25\right]^{-1/2} f_5\left(\frac{x^5}{x^4 + 5x^3 + 15x^2 + 25x + 25}\right).$$

Claim:

$$f_5(z) = Hl\left(\tfrac{-11\mp 2\mathrm{i}}{-11\pm 2\mathrm{i}}, -\tfrac{1}{50}(-11\mp 2\mathrm{i});\ \tfrac{1}{2}, \tfrac{1}{2}, 1;\ \underline{z/\left[-11\pm 2\mathrm{i}\right]}\right).$$
$$= \left[\tfrac{1}{5}(z^2 + 10z + 5)\right]^{-1/4} {}_2F_1\left(\tfrac{1}{12}, \tfrac{5}{12}; 1; \underline{12^3 z/(z^2 + 10z + 5)^3}\right).$$

#3: Combinatorial Functional Equations $[X_0(12), X_0(18)/X_0(6)]$

Let $F = F(z) = \sum_{n=0}^{\infty} a_n z^n$ be the generating function of the Franel numbers

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^3, \qquad n \geq 0.$$

Then F, which is defined on the disk $|z| < 1/8$, satisfies the quadratic and cubic functional equations

$$F\left(\frac{x(x+6)}{8(x+3)^2}\right) = 2\left[\frac{x+3}{x+6}\right] F\left(\frac{x^2}{8(x+3)(x+6)}\right),$$

$$F\left(\frac{x(x^2+6x+12)}{8(x+3)(x^2+3x+3)}\right) = 3\left[\frac{x^2+3x+3}{(x+3)^2}\right] F\left(\frac{x^3}{8(x+3)^3}\right),$$

for $|x|$ sufficiently small, and also for all $x > 0$.

Claim: $F(z) = Hl(-8, -2; 1, 1, 1, 1; 8z) = Hl(-\frac{1}{8}, \frac{1}{4}; 1, 1, 1, 1; -z)$.

The Algebro–Geometric Infrastructure

Elliptic Curves over $\mathbb{C}$

- Any $E/\mathbb{C}$
  - ◇ has a projective Weierstraß model, the affine portion of which is

$$y^2 = 4x^3 - g_2 x - g_3$$

    in $\mathbb{C}^2 \ni (x, y)$, parametrized by $g_2, g_3 \in \mathbb{C}$ (not both zero).
  - ◇ has periods $\tau_1, \tau_2 \in \mathbb{C} \setminus \{0\}$, and period ratio $\tau := \tau_1/\tau_2 \in \mathbb{H}$.

- Any two E, E' are isomorphic iff their period ratios τ, τ' are related by some $g \in \Gamma(1) := PSL(2, \mathbb{Z})$, i.e.,

$$\tau' = (a\tau + b)/(c\tau + d), \qquad a, b, c, d \in \mathbb{Z}, \qquad ad - bc = 1.$$

The Universal Family $\mathcal{E}_1 \to X(1)$

- The moduli space of elliptic curves over $\mathbb{C}$ up to isomorphism is $Y(1) := \Gamma(1) \setminus \mathbb{H}$, with natural compactification $X(1) := \Gamma(1) \setminus \left[\mathbb{H}^* = \mathbb{H} \cup \left(\mathbb{Q} \cup \{\mathrm{i}\infty\} = \mathbb{P}^1(\mathbb{Q})\right)\right]$.
  - ◇ g_2, g_3 are (multivalued!) functions on $X(1)$.
- The modular curve $X(1)$ is of genus zero: $X(1) \cong \mathbb{P}^1(\mathbb{C})_j$, where j is a Hauptmodul, e.g., the Klein invariant $j := 12^3\, g_2^3/(g_2^3 - 27g_3^2)$.
- Isomorphism classes of elliptic curves are bijective with $\mathbb{P}^1(\mathbb{C})_j \setminus \{\infty\}$. So, there is a *universal family* of elliptic curves: $\mathcal{E}_1 \xrightarrow{\pi} X(1)$.
  - ◇ The fibre above $j = 0$ is *equianharmonic*: $g_2 = 0$, e.g., $\tau = \zeta_3$.
  - ◇ The fibre above $j = 12^3$ is *lemniscatic*: $g_3 = 0$, e.g., $\tau = \mathrm{i}$.
  - ◇ The fibre above $j = \infty$ is singular, e.g., $\tau = \mathrm{i}\infty$.

For Concreteness: q-Series

Near $\tau = \mathrm{i}\infty$ on $X(1) = \Gamma(1) \setminus \mathbb{H}^*$, one can expand in $q := e^{2\pi \mathrm{i}\tau}$, where $0 < |q| < 1$ corresponds to $\tau \in \mathbb{H}$. [Generators $\tau \mapsto \tau + 1$, $\tau \mapsto -1/\tau$ of $\Gamma(1)$ correspond to $q \mapsto q$, $q \mapsto \exp(4\pi^2/\log q)$.]

- The j-invariant: $j = q^{-1} + 744 + O(q^1)$.

- The invariants g_2, g_3 ("Eisenstein sums"):

  $g_2 \propto 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n$, $\qquad g_3 \propto 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n$.

- The Dedekind eta function:

$$\eta(\tau) := q^{1/24} \prod_{m=1}^{\infty} (1 - q^m).$$

Modular Forms and Functions

An entire function $f : \mathbb{H} \to \mathbb{P}^1(\mathbb{C})$ is said to be a modular form on a subgroup $G \leqslant \Gamma(1)$, of weight k, if

$$f((a\tau + b)/(c\tau + d) = \chi(a, b, c, d) \cdot (c\tau + d)^k f(\tau)$$

for all $g = \pm \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$, with $c > 0$.
Here $\chi : G \to U(1)$ is a character, e.g., a Dirichlet one (depends only on d).

- j is modular of weight 0, i.e., a modular *function*. (χ is trivial.)
- g_2 is modular of weight 4. (χ is trivial.)
- g_3 is modular of weight 6. (χ is trivial.)
- η is modular of weight $1/2$. (χ is complicated.)

Modular Subgroups: $\Gamma_0(N) < \Gamma(1)$

If $g_N := \left(\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix}\right)$, i.e., g_N is the N-isogeny $\tau \mapsto N\tau$, then

- $j = j(\tau)$ is stable under $\Gamma(1)$,
  so j is a Hauptmodul for $X(1) = \Gamma(1) \setminus \mathbb{H}^*$;

- $j' = j'(\tau) := j(N\tau)$ is stable under $\Gamma(1)' := g_N \Gamma(1) g_N^{-1} < PSL(2, \mathbb{R})$,
  so j' is a Hauptmodul for $X(1)' = \Gamma(1)' \setminus \mathbb{H}^*$;

- j, j' are in the function field of $X_0(N) := \mathbb{H}^* \setminus \Gamma_0(N)$, where
  $\Gamma_0(N) := \Gamma(1) \cap \Gamma(1)' = \{g \in \Gamma(1) : c \equiv 0 \pmod{N}\}$.

Assertion: j, j' in fact *generate* the function field of $X_0(N)$,
which classifies elliptic curves (up to isomorphism), *plus N-isogenies.*

Coverings $X_0(N)/X(1)$. [Refs.: Schoeneberg, McKean & Moll.]

The covering map $X_0(N) \to X(1)$, induced by $\Gamma_0(N) < \Gamma(1)$,

- is a $\psi(N)$-sheeted covering, where $\psi(N) := N\prod_{p|N}\left(1+\frac{1}{p}\right)$.
- is branched *only* over $j = 0, 12^3, \infty$, with known branching structure.

So if $X_0(N)$ like $X(1)$ is a genus-zero complex curve, then

- the function field of $X_0(N)$ is generated by a Hauptmodul x_N, and $X_0(N) \cong \mathbb{P}^1(\mathbb{C})_{x_N}$.
- $j = j(x_N)$ is a degree-$\psi(N)$ rational function, with known branching structure.

  [The cusps of $X_0(N)$ are the points mapped to $j = \infty$ (i.e., $\tau = \mathrm{i}\infty$).]

The Hauptmoduls x_N, $N \geqslant 2$

Claim: For each $N \geqslant 2$ for which $X_0(N)$ is of genus zero, i.e.,

$$N = 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 18, 25,$$

a Hauptmodul x_N may be constructed as an eta quotient, e.g.,

$$x_4 = 2^8 \cdot [4]^8/[1]^8 := 2^8 \cdot \eta(4\tau)^8/\eta(\tau)^8,$$
$$\tilde{x}_4 = x_4/(x_4 + 16) = 2^4 \cdot [1]^8[4]^{16}/[2]^{24}.$$

Pedestrian Verification:

① Verify invariance of the alleged x_N under $\Gamma_0(N)$.

② Show the alleged x_N has exactly one zero and one pole on $X_0(N)$.
(Each can be chosen to lie at a cusp.)

Canonical Hauptmoduls as Eta Quotients

N	$x_N(\tau) = \kappa_N \cdot \hat{x}_N(\tau)$
2	$2^{12} \cdot [2]^{24} / \, [1]^{24}$
3	$3^{6} \cdot [3]^{12} / \, [1]^{12}$
4	$2^{8} \cdot [4]^{8} / \, [1]^{8}$
5	$5^{3} \cdot [5]^{6} / \, [1]^{6}$
6	$2^{3}3^{2} \cdot [2][6]^{5} / \, [1]^{5}[3]$
7	$7^{2} \cdot [7]^{4} / \, [1]^{4}$
8	$2^{5} \cdot [2]^{2}[8]^{4} / \, [1]^{4}[4]^{2}$
9	$3^{3} \cdot [9]^{3} / \, [1]^{3}$
10	$2^{2}5 \cdot [2]\,[10]^{3} / \, [1]^{3}[5]$
12	$2^{2}3 \cdot [2]^{2}[3]\,[12]^{3} / \, [1]^{3}[4]\,[6]^{2}$
13	$13 \cdot [13]^{2} / \, [1]^{2}$
16	$2^{3} \cdot [2]\,[16]^{2} / \, [1]^{2}[8]$
18	$2 \cdot 3 \cdot [2]\,[3]\,[18]^{2} / \, [1]^{2}[6]\,[9]$
25	$5 \cdot [25] \,/\, [1]$

Covering Maps $X_0(N) \to X(1)$

N	$j(\tau)$ as a function of $x_N(\tau)$
2	$\frac{(x+16)^3}{x} = 12^3 + \frac{(x+64)(x-8)^2}{x}$
3	$\frac{(x+27)(x+3)^3}{x} = 12^3 + \frac{(x^2+18x-27)^2}{x}$
4	$\frac{(x^2+16x+16)^3}{x(x+16)} = 12^3 + \frac{(x+8)^2(x^2+16x-8)^2}{x(x+16)}$
5	$\frac{(x^2+10x+5)^3}{x} = 12^3 + \frac{(x^2+22x+125)(x^2+4x-1)^2}{x}$
6	$\frac{(x+6)^3(x^3+18x^2+84x+24)^3}{x(x+8)^3(x+9)^2} = 12^3 + \frac{(x^2+12x+24)^2(x^4+24x^3+192x^2+504x-72)^2}{x(x+8)^3(x+9)^2}$

The Dual Covering Maps

N	$j'(\tau) := j(N\tau)$ as a function of $x_N(\tau)$
2	$\frac{(x+256)^3}{x^2} = 12^3 + \frac{(x+64)(x-512)^2}{x^2}$
3	$\frac{(x+27)(x+243)^3}{x^3} = 12^3 + \frac{(x^2-486x-19683)^2}{x^3}$
4	$\frac{(x^2+256x+4096)^3}{x^4(x+16)} = 12^3 + \frac{(x+32)^2(x^2-512x-8192)^2}{x^4(x+16)}$
5	$\frac{(x^2+250x+3125)^3}{x^5} = 12^3 + \frac{(x^2+22x+125)(x^2-500x-15625)^2}{x^5}$
6	$\frac{(x+12)^3(x^3+252x^2+3888x+15552)^3}{x^6(x+8)^2(x+9)^3} = 12^3 + \frac{(x^2+36x+216)^2(x^4-504x^3-13824x^2-124416x-373248)^2}{x^6(x+8)^2(x+9)^3}$

The Elliptic Families $\mathcal{E}_N \to X_0(N)$

- For each $N \geqslant 2$, there is a fibration $\mathcal{E}_N \to X_0(N) \to X(1)$ where each fibre is, formally, an elliptic curve (iso. class), plus an N-isogeny.

- If the N-isogeny is forgotten, this becomes a conventional elliptic family. (A rational one, if $X_0(N)$ has genus zero.)
  - Any elliptic curve (iso. class) appears as $\psi(N)$ fibres of $\mathcal{E}_N$.
  - Singular fibres of $\mathcal{E}_N$ include those above *cusps* [points on $X_0(N)$ mapped to $j = \infty$ on $X(1)$], and *elliptic points* [points on $X_0(N)$ mapped to $j = 0$ and $j = 12^3$ on $X(1)$].

- A Weierstraß model: if $j = P^3(t)S(t)/R(t) = 12^3 + Q^2(t)T(t)/R(t)$ where $t := x_N$, then $\mathcal{E}_N \to X_0(N)$ has model $y^2 = 4x^3 - 3P(t)S(t)T(t)\,x - Q(t)S(t)T^2(t)$.
Cf. Herfurtner's 1991 classification of certain elliptic families.

Classical Modular Equations, Rationally Parametrized

Each Hauptmodul x_N, a parameter for $X_0(N)$, rationally parametrizes pairs of N-isogenous (iso. classes of) elliptic curves.
I.e., it parametrizes the order-N modular relation: the relation between the transcendental functions $j = j(\tau)$ and $j' = j(N\tau)$ on $\mathbb{H}$.

E.g., $N = 2$:

$$j = (x_2 + 16)^3/x_2, \qquad j' = (x_2 + 256)^3/x_2^3.$$

Rational parametrization of pairs of fibres works at higher levels too.
E.g., the order-2 modular equation for x_4, coming from $X_0(8)/X_0(4)$:

$$x_4(\tau) = [x_8(x_8+8)](\tau), \qquad x_4(2\tau) = \left[x_8^2/(x_8+4)\right](\tau).$$

The rational function on each r.h.s. is of degree 2 because $\psi(8)/\psi(4) = 2$ is the index of $\Gamma_0(8)$ in $\Gamma_0(4)$, so $X_0(8)/X_0(4)$ is 2-sheeted.

From Hauptmoduls to Modular Forms

Theorem. If $f = f(\tau)$ is a weight-k modular form on $\Gamma(1)$, with trivial character, then $f(N\tau)/f(\tau)$, which is a single-valued function on $X_0(N)$, will be of weight 0, i.e., an element of the function field of $X_0(N)$. So it must be a rational function of the Hauptmodul x_N.

Strengthened version. Even if the character is nontrivial, in 'nice' cases (e.g., if it is Dirichlet), the quotient $f(N\tau)/f(\tau)$ will be a *finite-valued* function on $X_0(N)$, i.e., an algebraic function of the Hauptmodul x_N.

Both of these extend to higher levels (to modular forms on genus-zero $\Gamma_0(M)$, yielding rational/algebraic functions of x_{NM}).

Bringing in the Differential Equations

Theorem (Stiller 1980s, et al.). Any weight-k modular form f on a genus-zero modular subgroup $\Gamma_0(N) \cong \mathbb{P}^1(\mathbb{C})_{x_N}$, with trivial character, viewed as a function of the Hauptmodul x_N, satisfies a homogeneous linear order-$(k+1)$ ODE: a *Fuchsian differential equation*.

A new perspective: independent variable=x_N, dependent variable=f.

Strengthened version. The same occurs for modular forms with 'nice' nontrivial characters; and even for certain non-form functions, such as *roots* of modular forms, which may not even be single-valued on $\mathbb{H}$.

An Example: g_2 and $g_2^{1/4}$

- $g_2 = g_2(\tau)$ is a weight-4 form on $\Gamma(1)$ and must satisfy an order-5 Fuchsian ODE "on $X(1)$", with independent variable j.

- The fourth root $g_2^{1/4}$ is *not* a weight-1 modular form, since it fails to be single-valued on $\mathbb{H} \ni \tau$. But it 'almost' is one: each of its branches satisfies an order-2 ODE. In particular,

$$g_2^{1/4}(\tau) = {}_2F_1\left(\tfrac{1}{12}, \tfrac{5}{12};\ 1;\ \hat{J}(\tau)\right),$$

  where $\hat{J} := 12^3/j$. As a function of an appropriate Hauptmodul, it is a Gauss hypergeometric function! (Dedekind; Stiller 1988.)

New Algebraic Hypergeometric Transformations

In consequence, for $N = 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 18, 25$, there is a rationally parametrized algebraic hypergeometric transformation

$$
\begin{aligned}
&{}_2F_1\left(\tfrac{1}{12}, \tfrac{5}{12};\ 1;\ 12^3/j'(x_N)\right) \\
&\qquad\qquad = \mathrm{PREFACTOR}(x_N) \cdot {}_2F_1\left(\tfrac{1}{12}, \tfrac{5}{12};\ 1;\ 12^3/j(x_N)\right),
\end{aligned}
$$

coming from $g_2^{1/4}(12^3/j'(x_N)) = \mathrm{PREFACTOR}(x_N) \cdot g_2^{1/4}(12^3/j(x_N))$.
(Abuse of notation here...).

The prefactor is in general *algebraic*, not rational.

Picard–Fuchs Equations and Modular Forms

- Suppose that
  - $\mathcal{E} \xrightarrow{\pi} X = \Gamma \setminus \mathbb{H}^*$ is an elliptic family, where $\Gamma < \Gamma(1) := PSL(2, \mathbb{Z})$.
  - $\omega = \omega(x)$ is a meromorphic family of 1-forms, and cycles (homology classes) γ_1, γ_2 are defined as (multivalued) functions of $x \in X$.

- Then (cf. Stienstra–Beukers)
  - the second-order P–F equation satisfied by the periods $\int_{\gamma_i} \omega(x)$ has a weight-1 modular form $f(x)$ for Γ among its solutions. It may have a nontrivial [even, non-Dirichlet] character.
  - The full solution space of the P–F equation is $(\mathbb{C}\tau(x) \oplus \mathbb{C})f(x)$.

The Cases $\Gamma = \Gamma(1)$ and $\Gamma = \Gamma_0(N)$

- If $\Gamma = \Gamma(1)$,
  - ◇ the associated weight-1 modular form f_1 is $g_2^{1/4}$. (Not actually single-valued.)
  - ◇ the associated P–F equation satisfied by $f_1 = f_1(\hat{J})$ is the GHE satisfied by ${}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \hat{J}\right)$. Here $\hat{J} := 12^3/j$.

- If $\Gamma = \Gamma_0(N)$,
  - ◇ the associated weight-1 form f_N could be taken to be f_1, but...
  - ◇ an associated P–F equation can be obtained by pulling back the GHE along $X_0(N) \to X(1)$.
    Result: a Fuchsian ODE with a singular point at each singular fibre.
    And placing it in 'normal form' is best:
    a GHE (if there are 3 singular fibres), a HE (if there are 4),...

The P–F Equation $\mathcal{L}_N f_N = 0$ for $\mathcal{E}_N \to X_0(N)$

N	Operator $\mathcal{L}_N$, where $x := x_N$
2	$D_x^2 + \left[\frac{1}{x} + \frac{1}{2(x+64)}\right] D_x + \frac{1}{16x(x+64)}$
3	$D_x^2 + \left[\frac{1}{x} + \frac{2}{3(x+27)}\right] D_x + \frac{1}{9x(x+27)}$
4	$D_x^2 + \left[\frac{1}{x} + \frac{1}{x+16}\right] D_x + \frac{1}{4x(x+16)}$
5	$D_x^2 + \left[\frac{1}{x} + \frac{x+11}{x^2+22x+125}\right] D_x + \frac{x+10}{4x(x^2+22x+125)}$
6	$D_x^2 + \left[\frac{1}{x} + \frac{1}{x+8} + \frac{1}{x+9}\right] D_x + \frac{x+6}{x(x+8)(x+9)}$
7	$D_x^2 + \left[\frac{1}{x} + \frac{4x+26}{3(x^2+13x+49)}\right] D_x + \frac{4x+21}{9x(x^2+13x+49)}$
8	$D_x^2 + \left[\frac{1}{x} + \frac{1}{x+4} + \frac{1}{x+8}\right] D_x + \frac{1}{x(x+8)}$
9	$D_x^2 + \left[\frac{1}{x} + \frac{2x+9}{x^2+9x+27}\right] D_x + \frac{x+3}{x(x^2+9x+27)}$

N.B.: The scalar P–F operator $\mathcal{L}_9$ is equivalent to
the 2×2 matrix P–F operator computed by Dwork (1964).

Canonical Weight-1 Modular Forms (with character)

N	$x_N(\tau)$	$f_N(x_N(\tau))$	$\mathrm{cond}(\chi_N)$
2	$2^{12} \cdot [2]^{24} / [1]^{24}$	$[1]^4 / [2]^2$	—
3	$3^6 \cdot [3]^{12} / [1]^{12}$	$[1]^3 / [3]$	—
4	$2^8 \cdot [4]^8 / [1]^8$	$[= f_2(x_2(\tau))]$	—
5	$5^3 \cdot [5]^6 / [1]^6$	$\left\{[1]^5 / [5]\right\}^{1/2}$	—
6	$2^3 3^2 \cdot [2][6]^5 / [1]^5 [3]$	$[1]^6 [6] / [2]^3 [3]^2$	3
7	$7^2 \cdot [7]^4 / [1]^4$	$\left\{[1]^7 / [7]\right\}^{1/3}$	—
8	$2^5 \cdot [2]^2 [8]^4 / [1]^4 [4]^2$	$[= f_2(x_2(\tau))]$	4
9	$3^3 \cdot [9]^3 / [1]^3$	$[= f_3(x_3(\tau))]$	3

N.B.: Fractional powers in f_5, f_7 are related to $\mathcal{E}_5, \mathcal{E}_7$ having a singular fibre of Kodaira type III, II, rather than just I_1 and I_5, I_7.

The General Theorem

Let $\Gamma < \Gamma(1) = PSL(2,\mathbb{Z})$, and choose $M > 1$.
Let $\Gamma' := g_M \Gamma g_M^{-1} < PSL(2,\mathbb{R})$, and let $\Gamma^{(M)} := \Gamma \cap \Gamma'$.

If $\Gamma, \Gamma^{(M)}$ are of genus zero, with Hauptmoduls $x, x^{(M)}$, then $x(\tau), x(M\tau)$ have rational representations $\phi(x^{(M)}(\tau))$, $\phi'(x^{(M)}(\tau))$, and...

If $f = f(x)$ is the canonical weight-1 modular form from the P–F equation for the elliptic family $\mathcal{E}_\Gamma \xrightarrow{\pi} \Gamma \setminus \mathbb{H}^*$, then

$$f\left(\phi(x^{(M)})\right) = \mathrm{PREFACTOR}(x^{(M)}) \cdot f\left(\phi'(x^{(M)})\right).$$

Example: $\Gamma = \Gamma_0(N)$, $\Gamma^{(M)} = \Gamma_0(MN)$.

In this way, every (genus-zero) covering $X_0(MN)/X(N)$ yields an algebraic transformation of a special function.

Algebraic Transformations: Examples #1, 2, 3

① $X_0(8)/X_0(4)$. Let $f = 2^4 \cdot [1]^8[4]^{16}/[2]^{24}$, a weight-$1$ form on $X_0(4)$; view it as a function of the *alternative H'modul* $\tilde{x}_4 = 2^4 \cdot [1]^8[4]^{16}/[2]^{24}$. This is simply the complete elliptic integral, $K_2 = K_2(\alpha)$! Parametrize the relation between $\tilde{x}_4(\tau), \tilde{x}_4(2\tau)$ by x_8 to get Identity #1.

② $X_0(25)/X_0(5)$. Let $f = f_5 = \{[1]/[5]\}^{1/2}$, a weight-$1$ form on $X_0(5)$; view it as a function of the Hauptmodul x_5. (This was the function "$f_5 = f_5(z)$".) Parametrize the relation between $x_5(\tau), x_5(5\tau)$ by x_{25} to get Identity #2.

③ $X_0(12)/X_0(6)$, $X_0(18)/X(6)$. Let $f = [2]^3[3]^6/[1]^2[6]^3$, a weight-$1$ form on $X_0(6)$; view it as a function of the alt. Hauptmodul $x_6/(x_6+9)$. (This was the generating function "$F = F(z)$".) Parametrize the relation between $x_6(\tau), x_6(2\tau)$ by x_{12} to get Identity #3a, etc.

Ramanujan's Elliptic Integrals

- Ramanujan's complete elliptic integral

$$\mathsf{K}_r(\alpha_r) \propto {}_2F_1(1/r, 1-1/r;\ 1;\ \alpha_r)$$

when $r = 2, 3, 4$, is associated with families $\mathcal{E}_4$, $\mathcal{E}_3$, $\mathcal{E}_2$ (respectively).

It is simply a canonical weight-1 modular form on the base curve, i.e., a period, written as a function of an (alternative) Hauptmodul.

- In consequence: many new algebraic transformations of K_3 and K_4, e.g.,

$$\mathsf{K}_4\left(\frac{x(x+4)^5}{(x^2+6x+4)^2(x^2+8x+20)}\right) = 5\left[\frac{x^2+6x+4}{x^2+30x+100}\right]^{1/2}\mathsf{K}_4\left(\frac{x^5(x+4)}{(x^2+8x+20)(x^2+30x+100)^2}\right)$$

which comes from $\mathcal{E}_{10} \to \mathcal{E}_2$, i.e., from $X_0(10)/X_0(2)$ or $\Gamma(10) < \Gamma(2)$.

Current and Future Work

- Treating more elliptic families.
  - $\mathcal{E}_\Gamma \xrightarrow{\pi} \Gamma \setminus \mathbb{H}^*$, where Γ is a general genus-zero congruence subgroup of $PSL(2,\mathbb{Z})$, other than an $X_0(N)$.
    (Classified by Cummins–Pauli.)
  - $\mathcal{E}_\Gamma \xrightarrow{\pi} \Gamma \setminus \mathbb{H}^*$, where Γ is a genus-zero *non-congruence* subgroup.
    (Not yet classified.)
  - Elliptic families that are not of this quotient form.
    (Cf. Herfurtner's classification, for 4 singular fibres.)

- Extending these computations to pencils of other algebraic varieties. (E.g., lattice-polarized K3 surfaces; cf. Doran.)

- Treating *multivariate* families, discovering (perhaps) new algebraic transformations of multivariate hypergeometric functions.