Morrison: CY-singularities

CY-mflds are used in string/M-compactif.

* depend on parameters $a_1, \dots, a_k$

$$\vec{a} \to \vec{a}_{lim}$$

limit is not Calabi-Yau

* parameter spaces

Examples:

(Reminder: CY is a cpct Riemannian mfld M^{2n}; metric has holonomy $SU(n) \subseteq SO(2n)$,

Known: $n \geq 3 \Rightarrow \exists!$ cplx str.

metric is Kähler

$$\{\text{metric}\} = \{\text{cplx-struct.}\} \times \{\text{Kähler metric}\}$$

local product

$\cap$ $H^2(X; \mathbb{R})$

/ Aut

$n = 2$ different)

limits: cpcts objects w/ singularities

local str.

Examples:

1) $\mathbb{C}^n/\mathbb{Z}_m$ where $m | n$

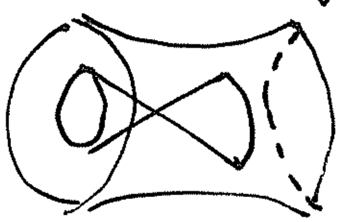
$$(x_1, \dots, x_n) \mapsto (e^{\frac{2\pi i}{m}} x_1, \dots, e^{\frac{2\pi i}{m}} x_n)$$

$dx_1 \wedge \dots \wedge dx_n$ is invariant

- $n = m = 2$: $\mathbb{C}^2/\mathbb{Z}_2$, first "rational double pts"

$$(x_1, x_2) \mapsto (-x_1, -x_2)$$

$$\underset{u}{\underset{\parallel}{x_1^2}}, \quad \underset{v}{\underset{\parallel}{x_1 x_2}}, \quad \underset{w}{\underset{\parallel}{x_2^2}}$$



$uw - v^2 = 0$

$\Downarrow$

deformed equ. $uw - v^2 = \varepsilon$

blowup: $\mathcal{O}(-2)$ over $\mathbb{CP}^1$

- $n = m = 3$ $\quad \mathbb{C}^3/\mathbb{Z}_3$

$$\frac{x_1^3, \dots, x_3^3}{u_1, \dots, u_{10}} \quad \text{relations}$$

<u>Thm</u>: (Schlessinger) $\mathbb{C}^n/\Gamma$, isolated singularities and $n \geq 3$

$\Longrightarrow \nexists$ cplx str. deformations

<u>Blowup singularity</u>

$$\begin{tikzcd} X \arrow[d, "\pi"'] \\ \mathbb{C}^n/\mathbb{Z}_m \end{tikzcd} \qquad \pi \text{ proper}$$

$\Downarrow$ $\quad \pi|_{X-\pi^{-1}(0)}$ is an isomorphism

π : X = total space of line bundle $\mathcal{O}(-m)$ over $\mathbb{C}P^{n-1}$

<u>$n = m = 3$</u> $\quad \mathcal{O}(-3)$ over $\mathbb{C}P^2$

<u>$n=4,\ m=2$</u> $\quad \mathcal{O}_{\mathbb{P}^3}(-2)$ has no non-vanish. holom. 4 - ~~[illegible]~~ form ;

$\mathbb{C}^4/\mathbb{Z}_2$ has no smooth blowup with non-vanishing 4 - form

$\mathbb{C}^4/\mathbb{Z}_k$ acts by $e^{2\pi i/k}$ $e^{-2\pi i/k}$

$e^{2\pi i a/k}$ $e^{-2\pi i a/k}$

$(a,k)=1$

* isolated singularities
* no toric blowup

More Examples:

2) $\{ x_1^k + x_2^k + \dots + x_{n+1}^k = 0 \}$

natural CY condition: $k \leq n$

$\mathbb{C}^3$: $k \leq 2$

$\mathbb{C}^4$: $k \leq 3$

$x_1^2 + \dots + x_4^2 = 0$

$x_1^3 + \dots + x_4^3 = 0$

$$\begin{array}{ccc} E & \subseteq & \tilde{X} \\ \downarrow & & \downarrow \\ p & \in & Y \end{array}$$

ω_Y = holom n-form on $Y - P$, no zeros

<u>CY</u>: $\pi^*(\omega_Y)$ is holomom. on one ($\Leftrightarrow$ on any) smooth blowup X

~~Thm: All~~

<u>Examples</u>: $x_1^3 + \dots + x_4^3 = 0$

- blow up is total of a line bundle over cubic surface in $\mathbb{C}P^3$

($\pi^*(\omega_Y)$ has no zeros)

$x_1^2 + \dots + x_4^2 = 0$

... quadric surface

$\pi^*(\omega_Y)$ vanishes

$x_1^2 + x_2^2 + \ldots + x_4^2$

$\Rightarrow (x_1 + i x_2)(x_1 - i x_2) + (x_3 + i x_4)(x_4 - i x_4)$

<u>Blowup</u> $\quad x_1 + i x_2 = x_3 + i x_4 = 0$

$\pi^{-1}(P) = \mathbb{CP}^1$

total space of $\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \mathcal{O}_{\mathbb{CP}}(-1)$

$\pi^*(\omega_Y)$ has no zeros

<u>Thm</u>: (Reid) $\quad x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$

has CY blowup if k even

has <u>no</u> CY blow up if K odd

A) <u>Cplx str</u>:

$n \geq 3 \implies H^{2,0}(X) = \{0\}$

$\implies H^2(X, \mathbb{R}) = H^{1,1}(X)$

$\lambda \in H^2(X; \mathbb{Z})$

$\lambda^2 > 0, \quad \lambda \cdot [C] > 0 \quad \forall$ holom $C \subseteq X$

$N\lambda \longleftrightarrow$ very ample line bundle on X

Fix $\alpha : X \hookrightarrow \mathbb{CP}^M$

$\mathcal{M}_{N,\lambda}\ \{ Z \subseteq \mathbb{CP}^M \}$ $\quad$ $\mathrm{hilb}.(Z) = \mathrm{hilb}(\alpha(X)) \Big/ \mathrm{Aut}(\mathbb{CP}^M)$

subschemes s

$\parallel$

parameter space

- take a GIT quotient
- for some bundles, not every nearly CY will have that bundle

$$\bigcup_{N,\lambda} \mathcal{M}_{N,\lambda}$$

$\cup\!\!|$

quasi-projective

" all # CYs of this type "

<u>K3 surfaces</u> $(n=2)$

restrict cplx. str to all those with

$H^{1,1}(X) \cap H^2(X;\mathbb{Z}) \supseteq L$ ← fixed lattice sign $(1, n-1)$

Kähler metrics to $L \otimes \mathbb{R}$

$\lambda \in L, N, \ldots$

$$\mathcal{U}\mathcal{M}_{N,\lambda} = \Gamma \backslash \overline{O(L^{\perp} \otimes \mathbb{R})} / K$$

periods + holom 2-forms

Quartics in $\mathbb{CP}^3$

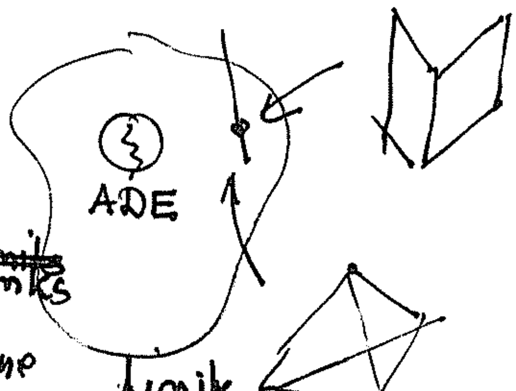
$x_1^2, x_2^2, \ldots x_4^2 = 0$ A

∞

2 ~~limits~~ kinks

Quartic + Quartic

Plane + Plane + Plane + Plane



finite monodromy cycles of

mirror sym $\sim$ // $\mathcal{M}_{\lambda,N}(\mathbb{Z})$

locally

Large cplx str. limit

B) Kähler structures

Kähler class is to be "complexified"

$$\{ \alpha \in H^2(X, \mathbb{C}/\mathbb{Z}) \mid \mathrm{Im}(\alpha) \text{ is Kähler} \} / \mathrm{Aut}(X)$$

$\mathrm{Im}(\alpha) \to \infty$

is a tube domain,

quotient is well-behaved (conjecturally)

(Aut(X) in Kähler cone)

{Kähler classes}

$n = 2, 3$

$\dim X = n$

$$\mathcal{K}(X) = \{\text{Kähler classes}\} \subseteq H^2(X;\mathbb{R})$$

$$\mathcal{K}(X) \subseteq \{\alpha \mid \textstyle\int \alpha^n > 0\}$$

$$\subseteq \{\alpha \mid \alpha^n > 0,\ \textstyle\int_C \alpha > 0 \ \forall \text{ holom } C \subset X\}$$

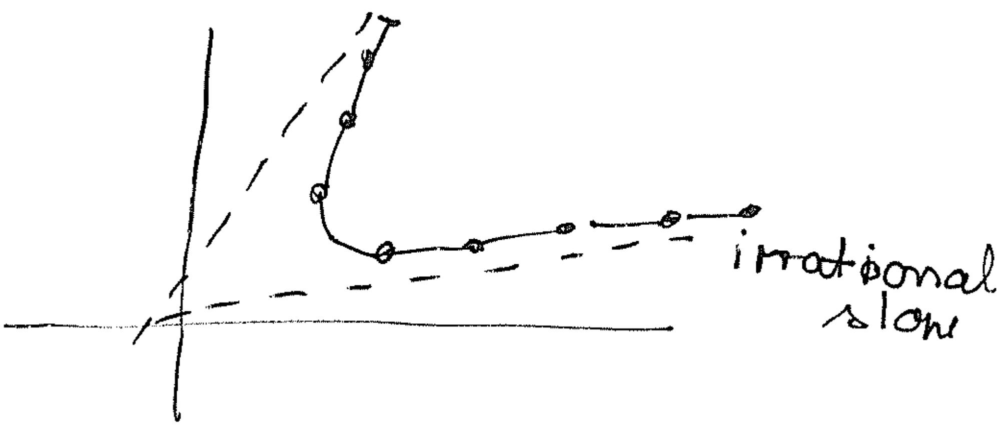
Kleiman's criterion:

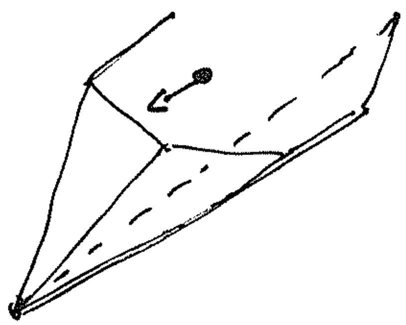
$$\mathcal{K}(X) = \operatorname{int}\left(\{\alpha \mid \alpha^n > 0,\ \textstyle\int_C \alpha > 0\}\right)$$

<u>$n \leq 3$:</u> $\overline{\mathcal{K}(X)}$ is locally polyhedral away from $\{\alpha \mid \alpha^n = 0\}$

(<u>$n = 2$:</u> $\mathcal{K}(X)$ is locally polyhedral)

slice: could look like





$$\int_C \alpha \lim = 0$$

$$\parallel$$

$$\lim_{\alpha \to \alpha_n} \operatorname{area}(C) = 0$$

$\underline{n=2}$

$C \longrightarrow pt$

$\Longrightarrow \quad C \cong \mathbb{CP}^1$

(higher codim mult rat. curves shrunk)

$\Longrightarrow$ ADE graph $\left(\begin{array}{l} \underline{n=2} \\ \cdot \text{ holom 2-form preserved} \\ \circ \end{array} \right)$

type II A string: non-abelian gauge symmetry $\Longleftrightarrow$ ADE grps

$\underline{n=3}$

0) every $x \in X$ belongs to C_x,
$\int_{C_x} \alpha \dim = 0$ $\quad$ ($X \xrightarrow{\text{collapses}}$ lower dim)

($0 = x_1 + i x_2 = x_3 + i x_4$)

1) finite set of curves collapses (to pts)

2) 1-param family of curves collapse to a curve $\quad \mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$

3) multi-param fam of curves all in a surface S, collapse to pts $\quad \mathbb{C}^3/\mathbb{Z}_3$