D. Morrison : Calabi - Yau - Sing. (1)

resolution is CY :

$$\begin{array}{c} \tilde{X} \\ \pi \downarrow \\ p \in X \end{array} \quad w/ \quad K_{\tilde{X}} \equiv \mathcal{O}$$

$X \setminus \mathrm{Sing}\, X \cong \tilde{X} \setminus \pi^{-1}(\mathrm{Sing})$

$\cong \tilde{X} \setminus \pi^{-1}(\mathrm{Sing}\, X)$

has holom. n-form

sheaf $(i_* \omega_{X \setminus \mathrm{Sing}\, X})^{\vee\vee}$ should be locally free of rank 1 w/ a nowhere-vanishing section

Tech cond : Cohen-Macaulay

dualizing complex is concentrated (ω_X° is concentrated in a single degree)

Gorenstein

"canonical", which means for any blow-up $\begin{array}{c} Y \\ \eta \downarrow \\ X \end{array}$

the $\eta^*((i_* \omega_X)^{\vee\vee})$ has at most zeros, no poles

<u>Def</u>: X has <u>canonical sing.</u> if it is Cohen-Macaulay, $\mathbb{Q}$-Gorenstein, and $\eta^*((i_* \omega_X)^{\vee\vee})$ or $\eta^*((i_* \omega_X^{\otimes N})^{\vee\vee})$ is regular for some (any) resolution of singularities.

<u>Question</u>: Does every Gorenstein canon singularity $p \in X$ have a resolution $\pi : \tilde{X} \to X$ s.t. $\pi^*(\omega_X^{\vee\vee})$ has no zeros near $\pi^{-1}(p)$

<u>Answer</u>: Yes in dim 2
No in dim > 2

In dim 3, the answer is "yes, unless ..."

dim 2

Gor. canon sing = rational double pts

$\mathbb{C}^2/G \quad G \subseteq SU(2)$ finite

$i_*(\omega_X)^{\vee\vee}$

in dim 2, there is a minimal resolution of sing,

minimal resolution of RDP is collection of E_i:

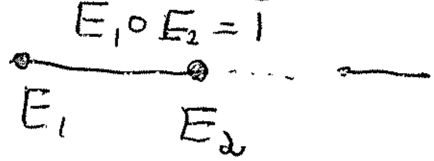
$$\begin{tikzcd} Y \arrow[d] \arrow[dd, bend right] \\ \tilde{X} \arrow[d] \\ X \end{tikzcd}$$

$E_i^2 = -2, \quad K_{\tilde{X}} \cdot E_i = 0$

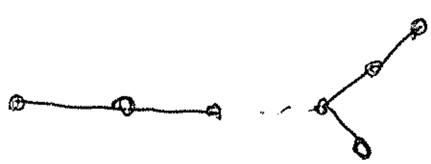
$i \neq j, \quad E_i \circ E_j \in \{0; 1\}$

$(E_i \circ E_j)_{ij}$ is neg definite

- simply laced Dykin diag.

A_n 

D_n 

E_n 

- finite group of $SU(2)$

$G = \mathbb{Z}_n$

$G = BD_{8K}$

$G = \mathbb{T}, \mathbb{O}, \mathbb{I}$

in $\mathbb{Z}_n$ case $\quad xy + z^n = 0$

$\vdots \qquad \vdots$

E_6 : $\quad x^2 + y^3 + z^4 = 0$

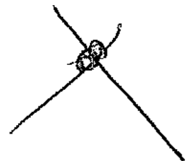
E_8 : $\quad x^2 + y^3 + z^5 = 0$

Bottom - up approach

$p \in X$, try to blow up in a way which doesn't introduce zeros into $\pi^*(\omega_X)^{\vee\vee}$

$xy + z^n = 0$ $\quad$ blow - up origin

$n \geq 2$ $\quad x_1 y_1 + z_1^{n-2} = 0$

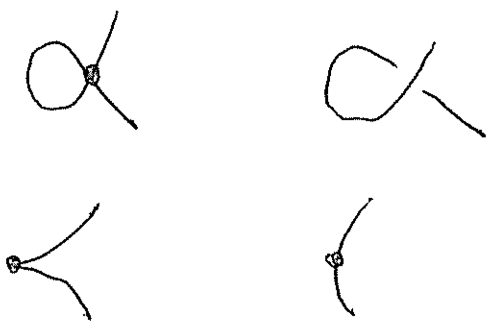


Blow up a rational double pt at p, result has no zeros of $\pi^*(\omega_X)^{\vee\vee}$ and only RDP's as blow up.

dim 3 : (Miles Reid 1980)

1) singularity might not be isolated

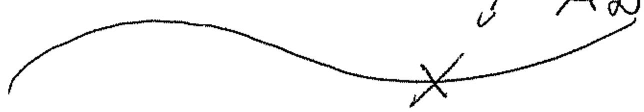
$$xy + x^3 + y^3 = 0$$



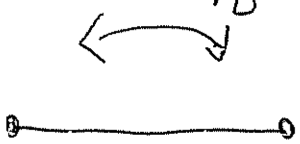
normalization (always introduces poles in section)

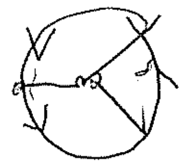
ADE classification

(+ monodromy



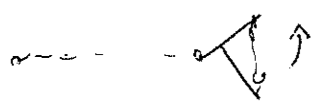
B_n



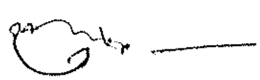


G

C_n :

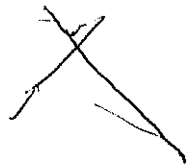


on



F_4

$xy + z^n \longrightarrow x_1 y + z^{n-2} = 0$



blow-up the ideal J to that component of $S_S X$ at generic pt, get partial res. of $R \tilde{\times} R$, monodromy invariant

Step 1: Resolve all curves of $R \tilde{\times} B$ by standard blow-ups.

Step 2: All remaining sing are isolated

Case 1: There is a blow-up

$$\begin{array}{c} \tilde{X} \\ \downarrow \pi \\ p \in X \end{array}$$

s.t. $\pi^{-1}(p)$ contains a divisor D on which $\pi^*(\omega_X)$ does not vanish.

Case 2: $\forall$ blow-up containing divisors, $\pi^*(\omega_X)$ vanishes along divisors

$xy + zu = 0$ $\qquad$ $x = z = 0$ $\qquad$ $x = u = 0$

Case 1: consider the tangent cone to X at p
$\subseteq$ Zariski tangent space to X at p

and "projectivize" (weighted project. if $d = 1$ or 2)

(del Pezzo surface) $\subseteq \mathbb{P}\,\mathrm{Zar}(X)$

(might be reducible, might have singularities RDP or simple elliptic might be non-normal)

S. s.t. $-K_S$ is ample

$1 \leq \deg S = (-K_S)^2 \leq 9$

Warning: After 1 such blow-up there may be (again) be non-isolated sing.

Step 2, case 2:

Reid shows: The general surface S through P has a rational double singularity (cDV) "terminal"

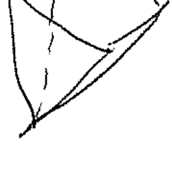
Some of these admit local CY resolutions (like small blow-ups) some do not:

$$x^2 + y^2 + z^2 + t^k = 0 \quad \text{terminal} \checkmark$$

has CY-resolution $\Leftrightarrow$ k even

Y = (non-sing) CY-mfld

K = Kähler cone



$[\alpha] \to \partial K$

expect a family of metrics in which $Z \subseteq Y$ shrinks to zero size leaving X = singular CY

$[\alpha] \rightarrow$ generic pts of ∂K

~~$\exists$ finite set $\gamma_i \in H_2$~~

$\exists$ single $\gamma \in H_2$, $\gamma = [C]$

$\int_\gamma \alpha \rightarrow 0$ in ∂K

$[C_t] = \gamma$

$t \in \begin{cases} \text{finite set,} \\ \text{alg. curve,} \\ \text{dim} \geq 2, \end{cases}$

$P, Q \in Y \rightarrow \pi(P) - \pi(Q) \in X$

$\Longleftrightarrow P, Q \in C$

$[C] = \gamma$

1. finite set : $Y \supseteq C_i \rightarrow P_i \in X$
2. $\{C_t\} = E \subseteq Y \rightarrow \Gamma \subseteq X$

$\downarrow$

Γ

3. $\{C_t\} = D \subseteq Y$

$\downarrow$

$p \in X$

L

Literature:

- 1st half: M. Reid,
  Canonical 3-folds, Angers 1979

- P. Wilson, Inv. 107 (1992) 561-585