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Calabi-Yau Singularities II

$$\begin{array}{c}\tilde{X}\\ \pi\downarrow\\ p\in X\end{array} \qquad K_{\tilde{X}} \equiv 0$$

$$X \setminus \mathrm{Sing}\, X \cong \tilde{X} \setminus \pi^{-1}(\mathrm{sing}\, X)$$

has a holo. n-form

double dual better behaved

Gorenstein

sheaf $(i_* \omega_{X \setminus \mathrm{Sing}\, X})^{\vee\vee}$ should be locally free of rank 1 w/ nowhere-vanishing section

Technical condition: Cohen-Macauly ($\omega_X^{\cdot}$ is concentrated in a single degree)

"canonical": $\forall$ blowup

$$\begin{array}{c}Y\\ \eta\downarrow\\ X\end{array}$$ the pullback $\eta^*((i_*\omega_X)^{\vee\vee})$ has at most only zeros, no poles

def X has canonical singularities if it is Cohen-Macaulay, $\mathbb{Q}$-Gorenstein, and $\eta^*((i_*\omega_X^{\otimes N})^{\vee\vee})$ is regular for some (any) resolution of singularities.

may be need a power

equivalent

Question Does every Gorenstein canonical singularity $p \in X$ have a resolution $\pi: \tilde{X} \to X$ s.t. $\pi^*(\omega_X^{\vee\vee})$ has no zeros near $\pi^{-1}(p)$?

Answer Yes in dim 2

No in dim > 2

In dim 3, answer is "yes, unless..."

dim 2

Gor. canon. sing = rational double pts

$\mathbb{C}^2/G$, $G \subseteq SU(2)$ finite

In dim 2, $\exists$ a well defined <u>minimal</u> resolution of singularities

$$\begin{tikzcd} Y \arrow[r] \arrow[dr] & \tilde{X} \arrow[d] \\ & X \end{tikzcd}$$

guaranteed $\exists$ trivial nbhd of singularity

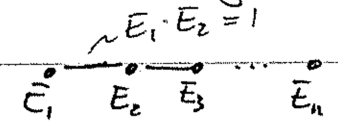
minimal resolution of RDP is a collection of E_i

$E_i^2 = -2$, $K_{\tilde{X}} \cdot E_i = 0$

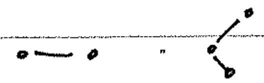
$i \neq j$, $E_i \cdot E_j \in \{0, 1\}$ $(E_i \cdot E_j)$ is negative definite

$2g - 2 = K \cdot C + C^2$ $\rightsquigarrow$ genus 0

$\rightsquigarrow$ simply laced Dynkin diagrams

A_n 

$G = \mathbb{Z}_n \longleftrightarrow A_{n-1}$

D_n  $G = BD_{8k} \longleftrightarrow D_\ell$

E_n  $n = 6, 7, 8$ $G = \mathbb{T}, \mathbb{O}, \mathbb{I} \longleftrightarrow E_6, E_7, E_8$

lowest degree 2

$\mathbb{Z}_n$: $xy + z^n = 0$

E_6: $x^2 + y^3 + z^4 = 0$

E_8: $x^2 + y^3 + z^5 = 0$

Bottom-up approach

$p \in X$, try to blow up in a way which does not introduce zeros into $\pi^*(\omega_X)^{\vee\vee}$

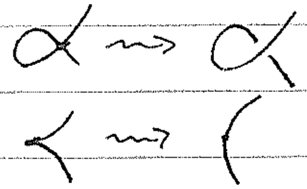
$$xy + z^n = 0 \quad (n \geq 2) \qquad \text{blow up origin:}$$

$$x_1 y_1 + z_1^{n-2} = 0$$

Blow up rational double pt @ p, result has no zeros of $\pi^*(\omega_X)^{\vee\vee}$ and only RDPs on blowup

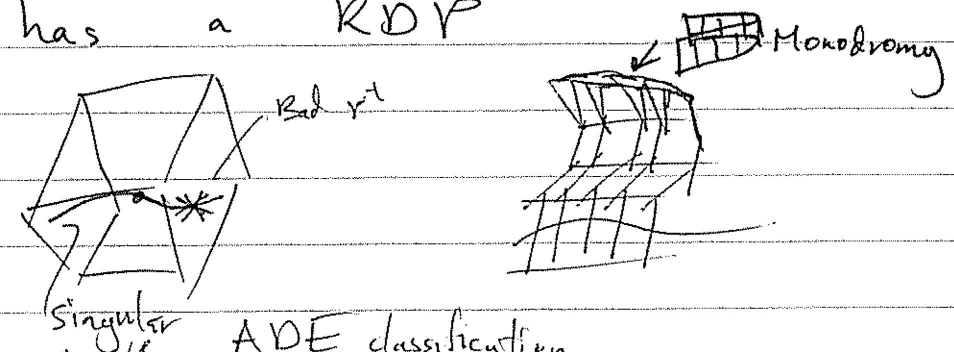
dim 3 (Miles Reid 1980)

① Singularity might not be isolated, but

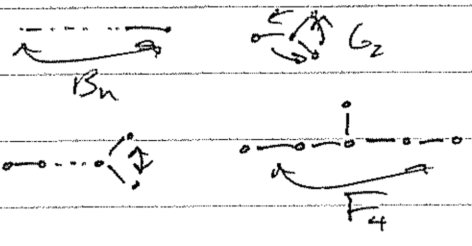


normalization (always introduces poles in codim 1)

the general surface trough curve of singularities has a RDP



ADE classification (+ monodromy)



$xy + z^n = 0 \longrightarrow x_1 y_1 + z_1^{n-2} = 0$

Blowup the ideal of that component of Sing X @ generic pt, get partial res. of RDP, monodromy invariant

Step 1: resolve all curves of RDPs by standard blowups

at most finite # of step 1, step 2, case 1, so that part terminates

(2) All remaining sings are isolated.

case 1) there is a blowup

$$\begin{array}{c} \tilde{X} \\ \pi \downarrow \\ p \in X \end{array}$$

s.t. $\pi^{-1}(p)$ contains a divisor D on which $\pi^*(\omega_X)$ does not vanish identically

case 2) $\forall$ blowups containing divisors, $\pi^*(\omega_X)$ vanishes along divisors

$xy + zu = 0$ $\quad x = z = 0$ or $x = u = 0$

(1) (2)

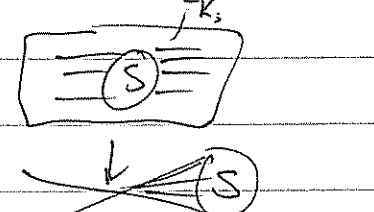
case 1) Consider the tangent cone to X at p $\subseteq$ Zariski tangent space to X at p. and "projectivize"? weighted if $\deg S = 1$ or 2

(del Pezzo surface) $\subseteq \mathbb{P}\,\mathrm{Zar}(X)$

S s.t. $-K_S$ is ample

(might be reducible, might have singularities, RDP or simple elliptic, might be non-normal)

$1 \le \deg S = (-K_S)^2 \le 9$



Warning: after 1 such blowup, may again be non-isolated sing!

Step 2 ~~case 2~~ | Reid shows: the general surface S through P has a rational double sing. (cDV)
"terminal"

some of these admit local CY resolution and some do not

$x^2 + y^2 + z^2 + t^k = 0$ terminal ✓

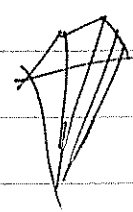
local CY resolution $\Leftrightarrow$ k even

this step doesn't introduce other sings ~~of other~~ types, so last step

Top Down

Y = (non-sing) CY mfld

K = Kähler cone



$[\alpha] \to \partial K$

expect a family of metrics in which $Z \subseteq Y$ shrinks to zero size, leaving X = singular CY

$[\alpha] \to$ generic pt of ∂K

∂K locally polyhedral, but may have accum. pts, don't know what to do there

$\exists$ single $\gamma \in H_2$, $\gamma = [C]$

$$\int_\gamma \alpha \to 0$$

$[C_t] = \gamma$ $\quad t \in \begin{cases} \text{finite set} \\ \text{alg. curve} \\ \text{dim} \geq 2 \end{cases}$

$P, Q \in Y \to \pi(P) = \pi(Q) \in X$

$\Longleftrightarrow$

$P, Q \in C$, $[C] = \gamma$

<u>Cases</u>

① $Y \supseteq C_i \longrightarrow p_i \in X$ like step 2 case 2

② $\{C_t\} = E \subseteq Y \longrightarrow \overline{\Gamma} \subseteq X$ step 1
$\downarrow$
Γ

③ $\{C_t\} = D \subseteq Y$ step 2 case 1
$\downarrow$
$p_t X$

Brandenbarger has thm about step 2 case 3, <u>coming soon</u>!

<u>References</u>

M. Reid, Canonical 3-folds, Angers (1979)
P. Wilson, ~~Invensia~~ 107 (1992) 561–583