3 April 2008
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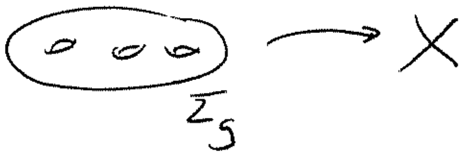
Quantum Curves and Random Matrices, II

invariants : moduli $t_1, \dots, t_n$
string coupling $\lambda = \hbar$

$$Z = \exp \sum_{g \geq 0} \lambda^{2g-2} F_g(t)$$

F_g :



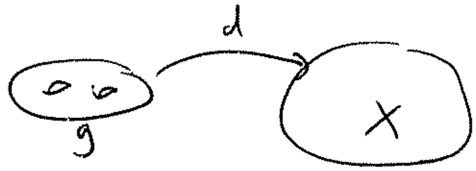
1. Random matrices

$$Z = \int_{N \times N} \mathcal{D}X \, e^{\frac{1}{\lambda} \mathrm{Tr}\, W(X)}$$

2. Gromov-Witten invariants of toric CY_3

$$F_g(t) = \sum_{d \geq 0} GW_{g,d} \, e^{-dt}$$

$GW_{g,d}$



$$\| \qquad \int_{[\mathcal{M}_g(X,d)]^{vir}} 1 \in \mathbb{Q}$$

$\lambda \approx 0$

3. Donaldson-Thomas invariants

$$Z = \sum_{d,n} DT_{d,n}\, e^{-n\lambda - dt}$$

sheaves of rank 1 on X

$rk = 1,\ d = c_2,\ n = ch_3$

$\lambda \to \infty$

4. Knot invariants

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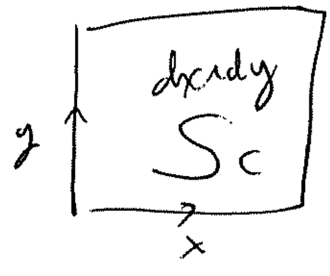
plane curves

$$B = E \times E' \simeq T^4$$

HK manifold, complex surface

def of B: $\quad B = \mathbb{C} \times \mathbb{C}, \quad B = \mathbb{C}^* \times \mathbb{C}^*$.

holo symplectic form $\quad \omega = dx \wedge dy, \quad \omega = \frac{dx}{x} \wedge \frac{dy}{y}$



$\mathbb{C}^2$

curve C: $F(x,y) = 0$.

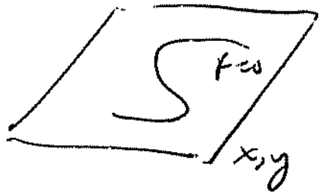
$\omega|_C = \partial \eta, \quad \eta$ holo 1-form

(C, η)

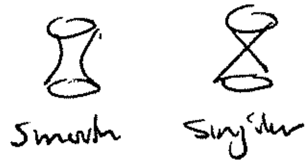
$B = \mathbb{C}^2$:

CY_3-fold hypersurface in $\mathbb{C}^4$

$$WZ + F(x,y) = 0.$$



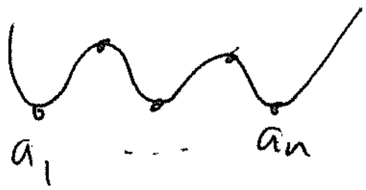
$WZ + F = 0$



$$\int DX \cdot \exp\left(\frac{1}{\lambda} \operatorname{Tr} W(X)\right) \qquad X = N \times N \text{ matrix}$$

W polynomial deg $n+1$.

$W'(a_i)$



$$X = U \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix} U^{-1}$$

$$N_i = \# \{ \text{e.v. } \lambda_I = a_i \}$$

$$t_i = N_i \lambda.$$

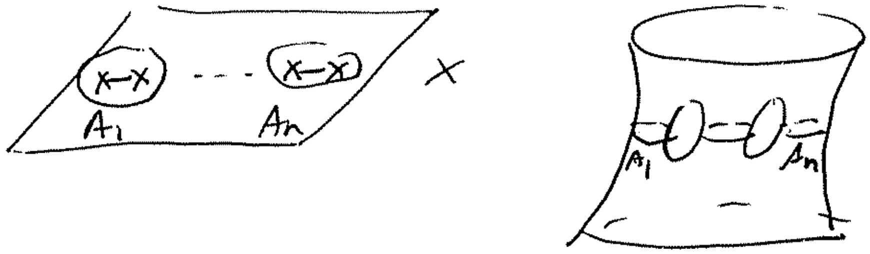
limit $N_i \to \infty$, $\lambda \to 0$, t_i fixed

$$N_1 + \cdots + N_n = N$$

<u>Thm</u> $Z \sim \exp\left(\sum_{g \geq 0} \lambda^{2g-2} F_g(t) \right)$

leading term $\mathcal{F}_0$

(spectra) curve C: $\quad y^2 = P_{2n}(x) = (W')^2 + f_{n-1}(x)$



$$\eta = y\,dx, \qquad \partial\eta = -dx \wedge dy.$$

$$\oint_{A_i} \eta = t_i \qquad \int_{B_i} \eta = \frac{\partial \mathcal{F}_0}{\partial t_i}$$

$$\Big\downarrow \text{determine}$$

$f_{n-1}(x)$.

$\mathcal{F}_g$?

$$\mathcal{F}_1 = -\frac{1}{2} \log \det \Delta_C$$

Riemann surface C

QFT of a chiral boson on C

$$Z = \int D\varphi \, e^{-\int \partial\varphi \wedge \bar\partial\varphi} = \frac{1}{\sqrt{\det \Delta_C}} = e^{\mathcal{F}_1}$$

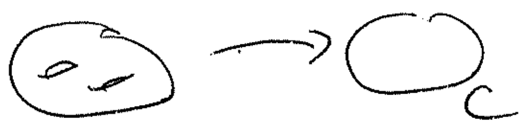
$$\varphi : C \to \mathbb{R}$$

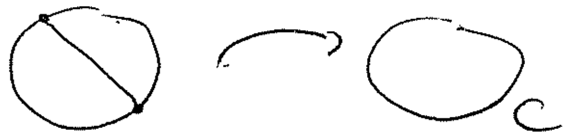
$$\int D\varphi \; e^{-\int \partial\varphi \wedge \bar\partial\varphi + A_{\bar z} \wedge \partial\varphi} = e^{\mathcal{F}_1 + \frac{1}{12}\mathcal{F}_0}$$

$$\varphi : C \to S^1 \qquad A = \frac{n}{2}$$

String theory

$\mathcal{F}_2$





these graphs are computing $\mathcal{F}_g$'s

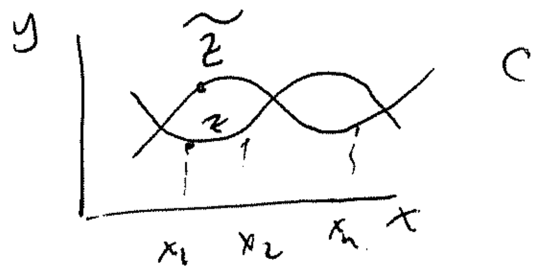
<u>Thm</u> [Eynard - Orantin]

$$\langle \mathcal{O} \rangle = \int \frac{dX \, e^{\frac{1}{\lambda} \mathrm{Tr} W(X)} \mathcal{O}}{Z}$$

$$W(x_1, \ldots, x_n) = \left\langle \mathrm{Tr} \frac{1}{x_1 - X} \cdots \mathrm{Tr} \frac{1}{x_n - X} \right\rangle^{\mathrm{con.}}$$

$$W = \sum_{g \geq 0} \lambda^{2g-2+n} W_g(x_1, \ldots, x_n)$$

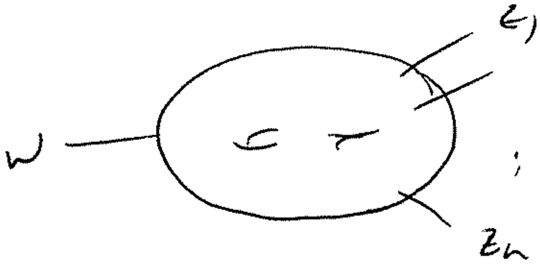
$$\lambda \left\langle \mathrm{Tr} \frac{1}{x - X} \right\rangle + W'(x) = \eta = y(x).$$



Bergmann kernel of C

$$B(z, w) \, dz \otimes dw = \langle \partial\varphi(z) \, \partial\varphi(w) \rangle \sim \frac{1}{(z-w)^2}$$

$$W_g(w, z_1, \ldots, z_n)$$

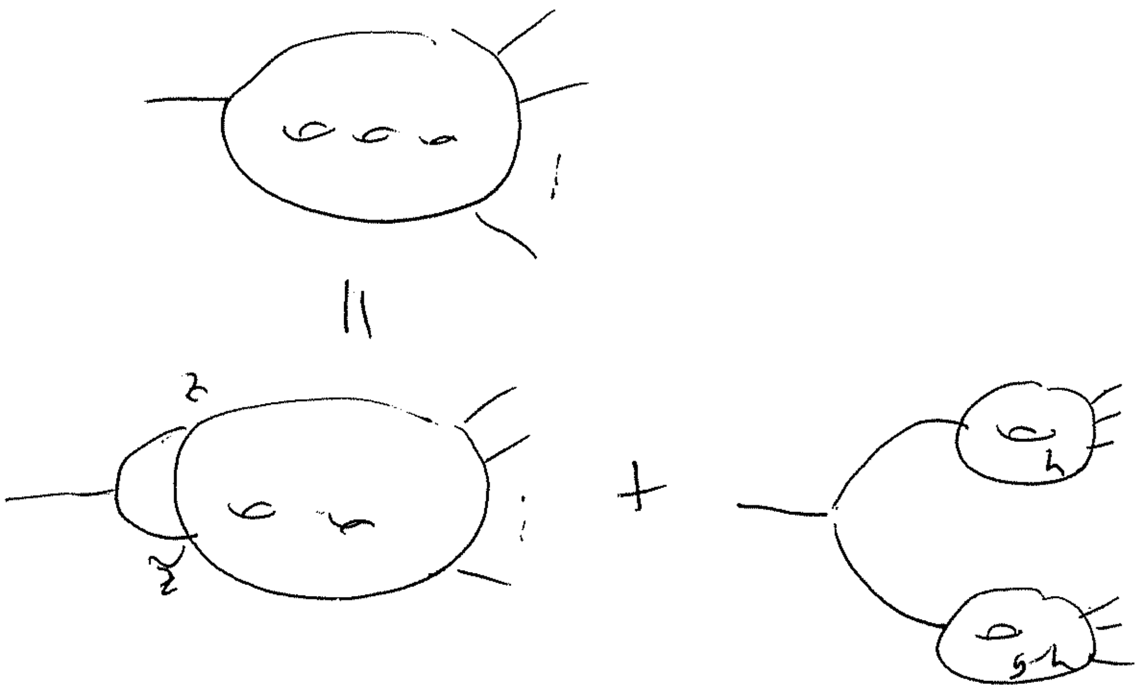


$$\| $$

$$\sum_{\substack{\text{branch pts} \\ p}} \mathrm{Res}_{z \to p} \left[\frac{\int_{\tilde{z}}^{z} B(v, w) \, dv}{\eta(z) - \eta(\tilde{z})} \left\{ W_{g-1}(z, \tilde{z}, z_1, \ldots, z_n) \right\} \right]$$

$+$ second term

$$\text{Second term} = \sum_{h=0}^{g} W_h(z, z, \ldots, z)\, W_{g-h}(\bar{z}, \ldots)$$



$$W_0(z, w) = B(z, w)$$

$$\eta = y\,dx = \partial \varphi_{cl}$$

$$g > 1: \quad F_g = \frac{-1}{2g-2} \sum_p \operatorname{Res}_{z=p} \left(W_g(z)\, \varphi_{cl}(z) \right)$$

C, $\bar\partial$, quantum theory of variations of complex structure

$\bar\partial_\mu = \bar\partial + \mu\partial$, μ = "Beltrami differential"

$$\mu = \mu_{\bar z}{}^{z} \frac{\partial}{\partial z} \otimes d\bar z$$

trivial variations $\mu = \bar\partial v$ $\quad v$ = holo. vector field $= v^z \frac{\partial}{\partial z}$

Variations that preserve $[\eta]$

$$\delta\eta = d\varphi$$

$$\delta\eta = \mathcal{L}_v \eta = d(\iota_v \eta) + \cancel{\iota_v d\eta}$$

$$\varphi = \iota_v \eta$$

$$v = \frac{\varphi}{\eta}$$

φ = chiral boson.

$$S = \int \partial\varphi \wedge \bar\partial\varphi$$

$$\bar\partial \to \bar\partial + \mu\partial \qquad \mu = \bar\partial v = \frac{\bar\partial\varphi}{\eta}$$

$$= \bar\partial + \frac{\bar\partial\varphi}{\eta}\partial$$

$$S = \int \partial\varphi\,\bar\partial\varphi + \frac{\bar\partial\varphi(\partial\varphi)^2}{\eta}$$

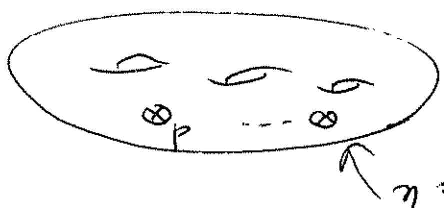
$$\int \mu T = \int \mu(\partial\varphi)^2 = \int \frac{\bar\partial\varphi(\partial\varphi)^2}{\eta}$$

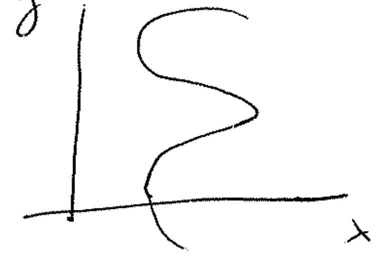
$$\int e^{-\int \partial\varphi\,\bar\partial\varphi + \lambda\frac{(\partial\varphi)^2\bar\partial\varphi}{\eta}} \qquad \partial\varphi(z_1) - \partial\varphi(z_n)$$

$$= TW(z_1, \ldots z_n)$$

$$\bar\partial\,\partial\varphi = 0.$$

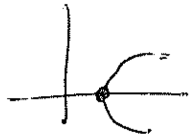
$$\int_C \frac{(\partial\varphi)^2\bar\partial\varphi}{\eta} = \int \bar\partial\left[\frac{(\partial\varphi)^2\varphi}{\eta}\right] = \sum\oint_p \frac{(\partial\varphi)^2\varphi}{\eta}$$



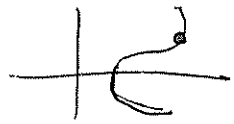


$\eta = y\,dx$

1) $y = 0$



2) $dx = 0$



1) local coordinate z $\qquad ydx \sim z\,dz$

$$\oint \frac{\varphi(\partial\varphi)^2}{z}\,dz\,|0\rangle = 0$$

$$\partial\varphi = \sum \alpha_n z^{-n-1}$$

$$\oint \sum_{n+m+k=-1} \frac{1}{n}\alpha_n\alpha_m\alpha_k|0\rangle$$

2) these <u>do</u> contribute

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chiral boson

$$\partial\varphi\,\bar\partial\varphi + \lambda\oint_{\substack{\text{branch pts}\\ P}} \frac{1}{\eta}\,\varphi(\partial\varphi)^2$$

<u>Claim</u>: Can derive recursion relations of Eynard–Orantin.

$$(C,\eta) \longrightarrow \mathcal{F}_g$$

chiral boson $\Longleftrightarrow$ chiral fermion

$$Z = \det \bar\partial_A$$

$C \qquad \boxed{\mathbb{Z}}\ (x,y)$

$\downarrow \lambda$

$[x,y] = \lambda$

curve $C \to$ D-module

$$\int DX\,DY\; e^{\mathrm{Tr}(XY + W(X) + V(Y))}$$

$$(y - W'(x))(x - V'(y)) + \dots = 0$$

$$y^p + x^q + \dots$$