4 April 2008
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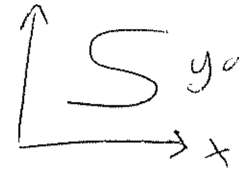
Quantum Curves and Random Matrices, II

$$\left.\begin{array}{l}\text{Matrix models}\\ \text{topological strings}\end{array}\right\} \longrightarrow \text{Spectral curves } F(x,y)=0.$$



$\int y\,dx$

$dx \wedge dy$
$\rightsquigarrow$ wave function

$$[x,y] = \hbar \Rightarrow y = -\hbar \frac{\partial}{\partial x} = -\hbar \partial$$

$$F(x,y) \rightsquigarrow \text{diff. op. } \hat{F}(x,\partial)$$

$$\hat{F}\psi = 0$$

spectral curve $\quad A(x)$ $n\times n$ matrix $\quad \det(y\,\mathbb{1} - A(x)) = F(x,y) = 0.$

$$\Rightarrow D_A = \frac{\partial}{\partial x} - A(x), \text{ rk } n. \qquad [D_A, \bar{D}] = 0$$

4d top. $N=4$ YM on M^4, HK, $U(N)$ gauge group.

$$Z = \sum_{p,n} d(n,p)\, e^{2\pi i(\vec{p}\cdot\vec{v} + \cancel{\ } (n - \frac{c}{24})\tau)}$$

Euler # of $\mathcal{M}^{inst}_{p,n} \in \mathbb{Z}$.

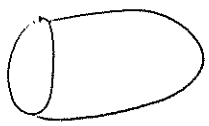
$$c_1 = \vec{p} \in H^2, \quad ch_2 = n$$

$$c = N\cdot\chi(M^4)$$

Z Jacobi form $\quad SL(2,\mathbb{Z}) \quad \tau \to \frac{a\tau+b}{c\tau+d}$

M^4 = ALE space $\quad \widetilde{\mathbb{C}^2/\Gamma} \quad \Gamma \subset SU(2), \ \Gamma = \mathbb{Z}_k$

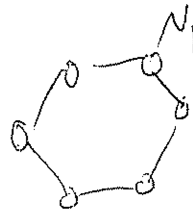
S^3/Γ



Pick flat connection, $\rho \in \mathrm{Hom}(\Gamma, U(N))$.

$\rho = \oplus N_i \rho_i \leftarrow$ irreps of Γ

$\mathbb{Z}_6$



McKay correspondence $\Gamma \leftrightarrow \hat{\mathfrak{g}}$

$\mathbb{Z}_k \quad SU(k)_N$

ρ dim $N \quad \hat{\rho}$

Claim: $Z(\widetilde{\mathbb{C}^2/\Gamma}; \tau, v) = \mathrm{Tr}_{\hat{\rho}}\left(e^{2\pi i \tau L_0} e^{2\pi i v J_0}\right)$

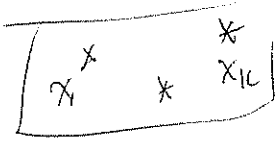
= character of rep $\hat{\rho}$ of $\hat{\mathfrak{g}}_N$.

$\Gamma = \mathbb{Z}_k, \ M^k \quad A_{k-1}$-sing $\subset TN_k$

$S^1 \to$

TN_k

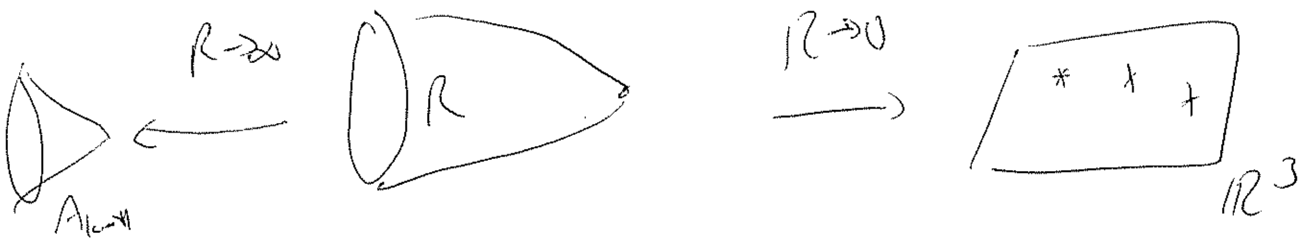
$\downarrow$



$\mathbb{R}^3$

$$ds^2 = H(d\vec{x})^2 + \frac{1}{H}(d\cancel{x}+\omega)^2$$

$$H = 1 + \sum_{i=1}^{k} \frac{1}{|\vec{x}-\vec{x}_i|}$$



D-branes

N D4-brane on $M^4 \times S^1$

lift to M-theory: $M^4 \times T^2$, N M5-branes

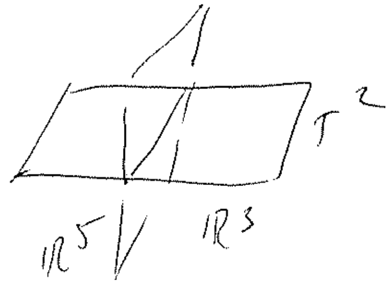
modulus of T^2 in τ

$B \times T^2$ — if $M^4 = TN$ then $B = \mathbb{R}^3$

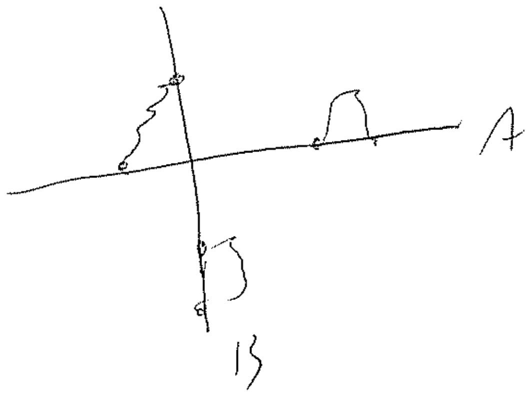
$\mathbb{R}^3 \times T^2$ — N D4-branes, with singularities

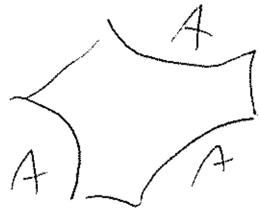
D6-brane at the singularities

U(N) gauge theory on TN_k $\cong$

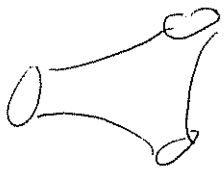


N D4-branes
k D6-branes
I-brane





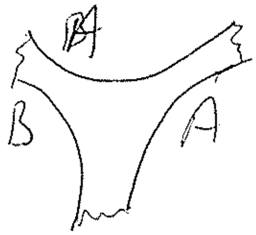
non-com. alg.



com. alg.

algebra $\mathcal{A}$, $\mathcal{B}$ $\Rightarrow$ gauge groups $U(N)$, $U(k)$

$\mathcal{M}$ A-B strings



$\mathcal{M}$ is an $\mathcal{A}$-$\mathcal{B}$ bimodule.

D0-D8 system
$\Rightarrow$ chiral fermions on T^2

$$\int d^2z \, \psi^{\dagger}_{i,a} \, \bar{\partial} \psi^{i,a} \qquad \begin{matrix} i=1,\dots,N \\ a=1,\dots,k \end{matrix}$$

Schur–Weyl duality $\quad U(N) \quad S_k$

$\mathcal{F} = (\mathbb{C}^N)^{\otimes k}$ decompose w.r.t. $U(N) \times S_k$.

$= \bigoplus V \otimes W$

irrep of $U(N)$ $\qquad$ irrep of S_k

Nk free fermions $\quad U(Nk)_1$

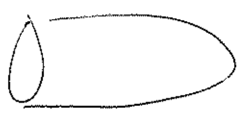
$$SU(N)_k \times SU(k)_N \times U(1)_{Nk} \subset U(Nk)_1$$

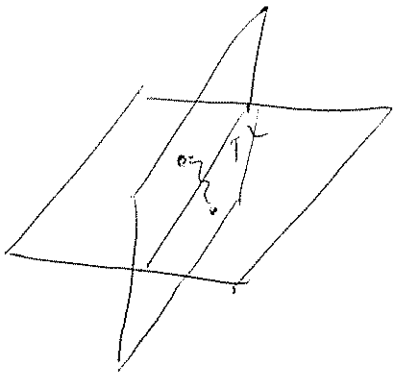
conformal embedding. $\mathcal{F}$ = free fermion module

$$\mathcal{F}^{\otimes Nk} = \bigoplus V_\rho \otimes W_{\rho^T}$$

irrep of $SU(N)_k$ $\qquad$ irrep of $SU(k)_N$

Nakajima $\quad \mathbb{C}^2/\mathbb{Z}_k \quad U(N) \longrightarrow SU(k)_N \times SU(N)_k$





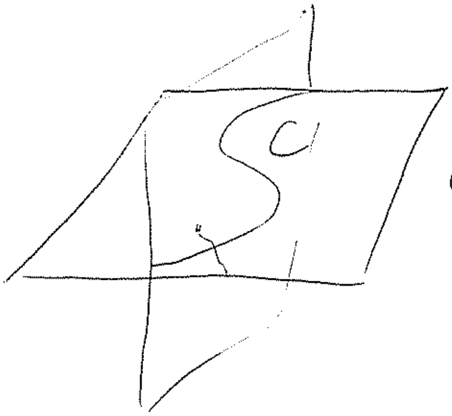
free fermions $N = k = 1$

$$Z = \det \bar{\partial}_{T^2}$$

$$= \frac{\sum_p e^{\pi i p^2 \tau + 2\pi i p v}}{\eta(\tau)}$$

where $\eta = \prod_{n>0} (1 - e^{2\pi i n \tau}) = \det^{1/2} \Delta$

$$= \sum_p e^{F_0 + F_1}$$



$\mathbb{C}^2 \Rightarrow$ free fermion on C

$\mathcal{N} = 2$ SW Solution $Z = \det \bar{\partial}_C$

Chain of dualities: string coupling constant $B = \frac{dx \wedge dy}{\lambda}$

$\mathcal{A} = \mathbb{C}[x,y] \xrightarrow{\beta\text{-field}} [x,y] = \lambda \qquad y \dashrightarrow \frac{\partial}{\partial x}$

$f(x,y)$

$\mathcal{A} = \langle x, \partial_x \rangle$ Sheaf of diff. ops on $\mathbb{C}$.

$D = \sum p_i(x)\partial^i$

intersecting strings $\psi(x)$

$\rightsquigarrow$ module for $\mathcal{A}$

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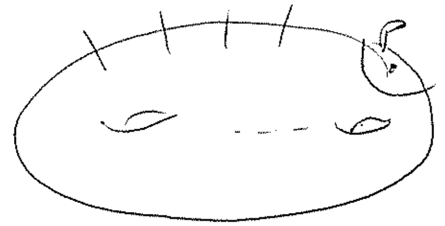
free fermions and tau-functions



$p = \{z = \infty\}$

$\det \bar{\partial}_C = Z.$

$$\psi(z) = \sum_{n \in \mathbb{Z}+\frac{1}{2}} \psi_n z^{-n-\frac{1}{2}} (dz)^{1/2}$$

$\{\psi_n, \psi_m^+\} = \delta_{n+m,0}$

$\psi^+(z)$

$\mathcal{F} = \prod \psi_{-n} \prod \psi_{-n}^+ |0\rangle$

$|0\rangle \qquad \psi_n|0\rangle = \psi_n^+|0\rangle = 0, \; n > 0$

$\psi(z) \in K^{1/2}$

$f \in K^{1/2}$

$\psi[f] = \oint f(z)\psi(z) \quad ; \; f(z) = z^{n+1}, \; \psi[z^{m}] = \psi_n$

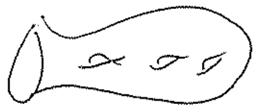
$|C\rangle: \quad \psi[f]\,|C\rangle = 0 \iff f \in H^0(C-p)$

$\mathbb{P}^1$

disk

regular z^n $\quad n>0$

$\rightsquigarrow |0\rangle$ vacuum

C 

$$\det \bar\partial_C = Z = \langle 0|C\rangle$$

$$|0\rangle, |C\rangle \in \mathcal{F}.$$

$$|t\rangle = \exp \sum t_n \alpha_{-n} |0\rangle \qquad \alpha_{-n} = \sum \psi_k^+ \psi_{k-n}$$

$$\det \bar\partial_t = \langle t|C\rangle$$

$\langle t|W\rangle = \tau(t)$ $\quad$ τ-function of the KP-hierarchy for any $|W\rangle \in \mathcal{F}$.

if $|W\rangle = |C\rangle \to$ geometric solution (Krichever) $\Rightarrow \vartheta$-function

if we have a D-module $\mathcal{M}$,

$$\psi[f]\,|W\rangle = 0 \qquad f \in M.$$

$$y^2 = u(x) \quad \rightsquigarrow \quad (\partial^2 - u(x))\psi = 0$$

curve

module $\mathcal{M} = \{ D\psi ;\ D \text{ is a } \mathcal{D} = \langle x, \partial \rangle \}$

$$\left\{ \begin{matrix} \psi, \ x\psi, \ x^2\psi, \ \dots \\ \partial\psi, \ x\partial\psi, \ \dots \end{matrix} \right\} \text{ module } \mathcal{M}.$$

$$\partial^2\psi - \cancel{u(x)}\psi$$

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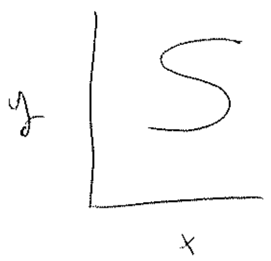
old matrix model

$$P = \sum p_i(x)\partial^i$$

$$Q = \sum q_i(x)\partial^i$$

$\mathbb{C}[x]$.

$$[P, Q] = 1$$



$$D = \partial_x + \underset{n \times n}{A(x)}$$

M^4

$$e^{F_1 \int_{M^4} \mathrm{Tr}\, R \wedge R}$$