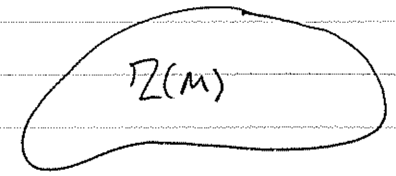
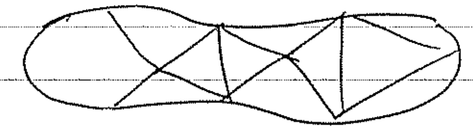
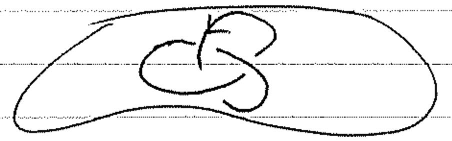
A Hyperbolic State-Sum Model for $SL(2,\mathbb{C})$ Chern-Simons

Tudor - Dimofte

Scribe: RDE







Wilson Loops

Witten - Chern-Simons and the Jones Poly

CS w/ $G \longleftrightarrow U_q(\mathfrak{g})$ Turaev-Viro

Simplicial triangula

$|Z(S)|^2$

Kurasteav., hyperbolic state sum

Hikami

1. CS Theory
2. Geometric Quantization
3. Hyperbolic Geometry
    Hikami's Invt
4. Correspondence

$$I[A] = \frac{k}{4\pi}\int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

A connection on principal G-bundle over M
Lie algebra valued 1-form

$$A \mapsto g^{-1} A g + g^{-1} dg$$

$SU(2)$ $\quad [T^a, T^b] = \varepsilon^{abc} T_c$

$$A = \sum_a A^a T^a \qquad \text{fields of theory}$$

Gauge inv up to shifts
$2\pi k \cdot n$

$$\int \mathcal{D}A \; e^{-iI[A]}$$

$SL(2,\mathbb{C})$ complexification of $SU(2)$

Classical solutions $\Longleftrightarrow$ Flat connections

$$F = dA + A \wedge A = 0$$

Aside:

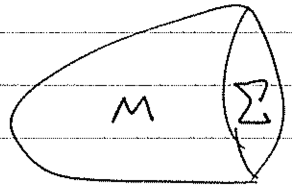
3D Quantum Gravity w/ negative cosmolgical constant Λ

$A = \omega + ie$ $\qquad \omega$ spin-connection
$\qquad e$ vielbein

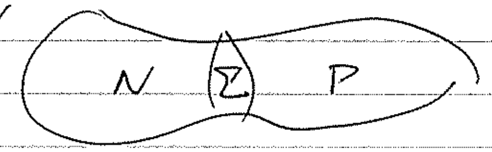
$$\hbar = \frac{\pi}{k}$$

$$Z_\hbar(M) = \sum_\alpha e^{\underbrace{1/\hbar\, S_0}_{\text{Classical action}} \underbrace{- \delta/2 \log \hbar + S_1}_{\text{Functional Det}} + \underbrace{\hbar S_2 + \ldots}_{\text{Higher Loop}}}$$

Want to compute $Z(M)$ on knot compliments



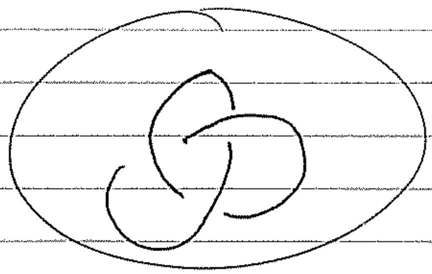
$Z(M)(\text{bdry}) \in \mathcal{H}_\Sigma$

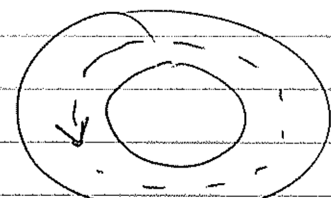
$Z\Big($ 

$\Big) = \langle Z(N) \mid Z(P) \rangle_{\mathcal{H}(\Sigma)}$

For $G = SU(2)$ $\quad \mathcal{H}(\Sigma)$ is finite dim

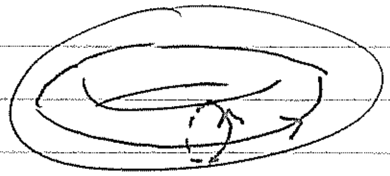
S^3



$Z\Big($  $\Big)$

two-torus w/ knot
$\Rightarrow \delta$ function

Geometric Quantization



$$\begin{pmatrix} l & * \\ 0 & 1/l \end{pmatrix} \quad \begin{pmatrix} m & * \\ 0 & 1/m \end{pmatrix}$$

$l, m \neq 0 \in \mathbb{C}$

u, v holonomies $\quad l = e^{v} \quad m = e^{u}$

Poisson Bracket $\{u, v\} = 1$

Classical Phase Space

$P = \{(u,v)\} \cong \mathbb{C}^* \times \mathbb{C}^*$

Promote to operators $\hat{u}, \hat{v}$

$$[\hat{u}, \hat{v}] = -i\hbar$$

$$\hat{u}\,\psi(u) = u\,\psi(u)$$
$$\hat{v}\,\psi(u) = i\hbar \frac{\partial}{\partial u}\psi(u)$$

$$\mathcal{H}_{T^2} = L^2(\mathbb{C}^*)$$

$(u_0, v_0) = P$

$f(u,v) = 0 \qquad$ (Lagrangian submanifold)

$\hat{f}(\hat{u}, \hat{v})\,\psi(u) = 0$

<u>Ex</u> QM on $\mathbb{R}$

p, q

Constraint fixed momentum

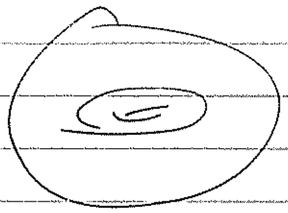
$$(\hat{p} - p_0)\psi(q) = 0 \Rightarrow \psi(q) = e^{i p_0/\hbar\, q}$$

$$(\hat{q} - q_0)\psi(q) = 0 \Rightarrow \psi(q) = \delta(q - q_0)$$

Can flat connection be extended to 3 mfld?

A polynomial of knot

$$A(\ell, m) = 0$$



$$A(\text{unknot}) = \ell - 1$$

3_1 $\quad A(\text{trefoil}) = (\ell-1)(1+m^6\ell)$

4_1 $\quad A(\text{figure-eight}) = (\ell-1)(1-(1-m^2-2m^4-m^6+m^8)\ell + \ell^2)$

solutions to $A(\ell, m)=0$ $\Longleftrightarrow$ # flat connections

classical ~~of~~ soln of CS

$$\hat{A}(\hat{\ell}, \hat{m})\, Z(M)(u) = 0$$

$$Z(M;u) = e^{1/\hbar\, S_0(u)} + \cdots$$

$$\hat{\ell} = e^{i\hbar\, d/du}$$

Get hierarchy of D.E.'s

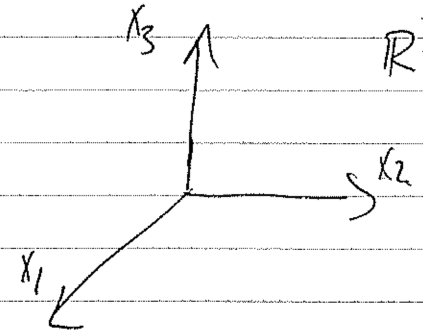
Only choice

$$A(e^{iS_0'(u)}, e^{u}) = 0$$

(Hyperbolic)

One set of flat connection $\Rightarrow$ $S_0^{geom} \approx \sim \frac{1}{2}\mathrm{Vol}(M)$

Hyperbolic Geometry



$\mathbb{R}^3$ $\quad ds^2 = \frac{1}{x_3^2}(dx_1^2 + dx_2^2 + dx_3^2)$

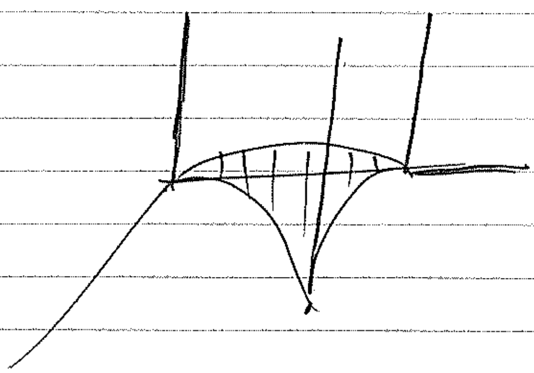
bdy Riemann sphere $\mathbb{R}^2 \cup \{\infty\}$

Isometries $PSL(2, \mathbb{C})$

Finite volume hyp 3 mfld

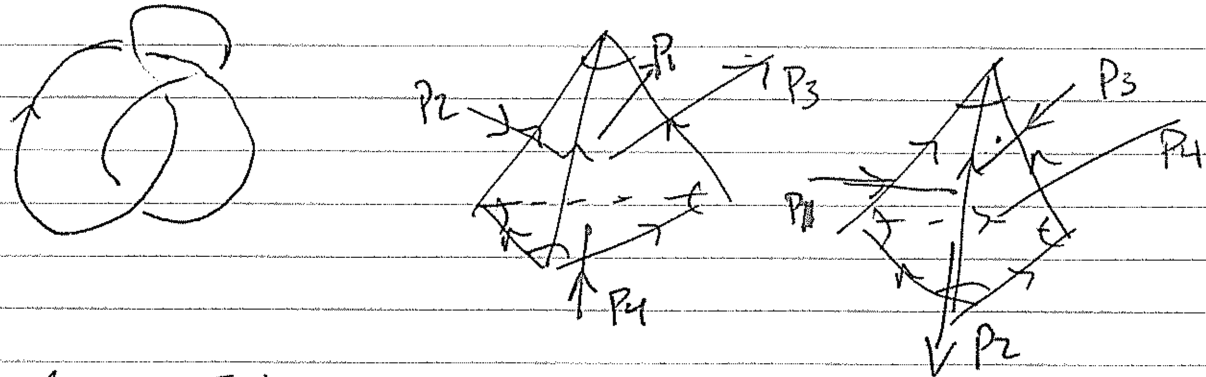
Knot compliment hyperbolic structure unless
Torus knot or satellite knot

Ideal tetrahedra
vertices lie on bdy of hyperbolic space

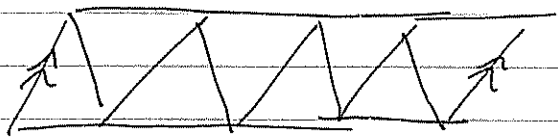


Conj: Every hyp 3mfld of finite volume
has a triangulation in terms of ideal tetrahedra

Figure Eight Knot



Cut out Eight corners



hyperbolic structures match $r(z, w) = 0$

$$\frac{z}{w} = m = e^{2u}$$

Hikami's Invt

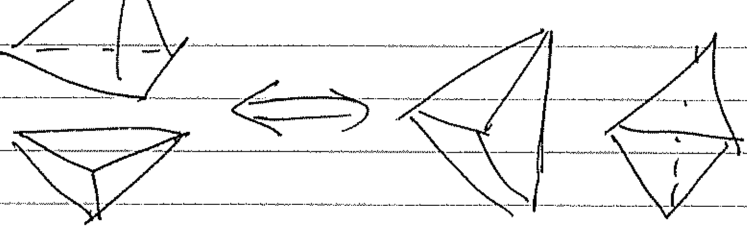
Triangulate

$$\Delta \sim V \otimes V \otimes V^* \otimes V^*$$

$$S_{ij}: V_i \otimes V_j \to V_i \otimes V_j$$

Pentagon Relation $S_{23} S_{12} = S_{12} S_{13} S_{23}$

Triangulations invt under 2-3 Pachner move



$f(p_1, p_2)$ $\quad S = e^{\frac{1}{2i\hbar}\hat{q}_1 \hat{p}_2} \, \Phi_\hbar(\hat{p}_1 + \hat{q}_2 - \hat{p}_2)$

$$\hat{q}_i \sim 2i\hbar \frac{d}{dp_i}$$

$$\Phi_\hbar(z) = \exp \frac{1}{4} \int_{\mathbb{R}+i\varepsilon} dx \, \frac{e^{-ixz}}{x \sinh(\pi x) \sinh(\hbar x)}$$

$$\langle p_1 \; p_3 \,|\, S \,|\, p_2 \; p_4 \rangle$$

$$\frac{\delta(p_1 + p_3 - p_2)}{\sqrt{4\pi\hbar}} \, \Phi_\hbar(p_4 - p_3 + i(\pi + \hbar))$$

$$\langle p_4 \;\; p_2 \,|\, S^{-1} \,|\, p_3 \; p_1 \rangle$$

$$\delta(\qquad) \, \frac{1}{\Phi_\hbar}$$

$$Z(M) = \int dp \; \Pi \langle \; \rangle \; \delta(\ldots u)$$

$$\delta(p_4 - p_3 - p_1 + p_2 = -2u) \left(\frac{z}{w}\right)^{-1}$$

Figure-Eight

$$= \frac{1}{4\pi\hbar} \int dp \, \frac{\Phi_\hbar(p + i\pi + i\hbar)}{\Phi_\hbar(-p - 2u - i\pi - i\hbar)} \; e^{?i/\hbar (u+p)}$$

$$U_q(sl_2)$$

$\sim e^{1/2\pi \, \mathrm{Vol}(M)}$ $\qquad \Phi_\hbar \sim \exp\left(\frac{1}{2\hbar} \mathrm{Vol}(\Delta)\right)$ $\qquad$ saddle point enforces gluing relation

$$\frac{1}{4\pi\hbar}\int dp \, \frac{\Phi_\hbar(p+i\pi+i\hbar)}{\Phi_\hbar(-p+2u-i\pi-\hbar)} \, e^{\frac{2i}{\hbar}u(u+p)}$$

$$= \sum_\beta e^{1/\hbar \, S_0^\beta - \frac{\delta^\beta}{2}\log\hbar}$$

Sum over saddle points

$$Z_\hbar^{cs\,\alpha}(u) = e^{\frac{-uc}{\hbar}} \, Z_\hbar^{\alpha}(u) \quad \text{Hikami}$$

$c = 2\pi$ for figure eight

constant $\rightsquigarrow S_1$ Ray-Singer Torsion

5_2 knot