The Geometric Langlands Conjecture I:

Arithmetic & Geometry
Abelian & non-Abelian

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* Arithmetic & Geometry
* Abelian Class Field Theory
* Geometric Langlands
* Hecke transforms
* Abelianization

next:

- QFT
- non-abelian Hodge theory
- Higgs bundles & Koszuls
- The classical limit

* GLC : • roots, chars, weights
• Bun, Loc, D^b

• Hecke correspondences, (affine) Grassmanians
• Hecke operators, automorphic sheaves
• stacks & gerbes

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* Examples: GL(n), GL(1).

• Higgs bundles, Hitchin system, abelianization
• Spectral & cameral covers

* Ancient approach: eigensheaves of abelianized heckes
• ⇝ duality between Higgs ^{L}Higgs

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* non abelian Hodge theory
• compact vs. non-compact
• stable, unstable, & wobbly bundles

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* Koszul resolutions of Higgs bundles
• Recent physics input

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* Classical limit: • definition
• conjecture
• theorem

proof:
• Duality along the base
• " " " fibers
• Transcendental ingredient: Hodge theory
• Duality for gerbes.

Arithmetic	Geometry
$\mathbb{Q}$	$\mathbb{P}^1$, $F(t)$ $\quad F = \mathbb{F}_q$, $q = p^n$
$\cap$	$\uparrow$
F finite extens	C, $F = F(C)$
	$\cup$
p prime $\cup\ \infty$	$x \in \mathbb{P}^1 = \mathbb{A}^1 \cup \infty$
"completions" $\mathbb{Q}_p$, $\mathbb{R}$	$\mathcal{O}_x \subset F_x$
$\mathbb{Z}_p \subset \mathbb{Q}_p$	
$\mathbb{A}_{\mathbb{Q}} \subset \prod_p \mathbb{Q}_p \times \mathbb{R}$	$\mathbb{A}_F \subset \ldots$

Abelian Class Field Theory

$$\mathrm{Gal}(F^{ab} \mid F) = \text{grp of connected comp.} \left(F^* \backslash \mathbb{A}_F^* \right)$$

$$\mathbb{A}_F^* = Gl_1(\mathbb{A}_F)$$

what for any reductive grp.

Langlands Conj

ℓ - adic rep's $\Longleftrightarrow$ automorphic fcts

$\sigma : Gal(\overline{F}/F) \to Gl_n(\overline{\mathbb{Q}_\ell})$ $\quad f : Gl_n(\mathbb{A}) \to \overline{\mathbb{Q}}_\ell$

$\ell \neq p$ $\quad$ bi-invariant

continuous $\quad$ $Gl_n(F) \backslash GL_n(\mathbb{A}) / Gl_n(\mathcal{O})$

nowhere ramified

geom. irreducible $\quad$ · cuspidal (new for n)

· Hecke eigenfcts

$\forall x \in C$, $0 \le i \le n$

$$(T_x^i f)(g) = \int_{h_x \in M_n^i \mathcal{O}_x} f(g h_x) \, dh_x$$

$g \in Gl_n(\mathbb{A})$

π_x : uniformizer at x

$$Gl_n(\mathcal{O}_x) \operatorname{diag}(\pi_x, 1, 1, \ldots, 1) \, Gl_n(\mathcal{O}_x)$$

Conj: For ~~$\forall \sigma$~~ ℓ-adic rep $\exists f_\sigma$

$$T_x^i f_\sigma = (\# m(x)) \operatorname{tr}(\Lambda^i \sigma(\mathrm{Frob}_x)) f_\sigma$$

Geometric Langlands replace functions (c) by sheaves

Grothendieck dict: sheaves $\xrightarrow[\mathrm{tr}(\mathrm{Frob})]{}$ function

Very geometric Langlands

analogue for Riem. surfaces

$n=1$: C any smooth cplx. Riem Surf.

"σ": rank 1 local sys on C

$\Longleftrightarrow$ (L, ∇) ∇ flat connection

$\ldots \backslash Gl_n(\mathbb{A})/\ldots = \mathrm{Bun}_n =$ moduli space of rank n VBs on C

(A. Weil)

$\mathrm{Bun}_1(C) =$ Picard Variety (C)

$\approx \mathbb{Z} \times J(C)$

g dim cplx torus
Jacobian

For $n=1$

any (L, σ) comes from a similar object on $\mathrm{Pic}(C)$

$$AJ: \quad C \longrightarrow \mathrm{Pic}(C)$$
$$x \longmapsto \mathcal{O}_C(x)$$

why AJ consider Hecke transforms

$V, x, i \overset{\text{corresponds}}{\rightsquigarrow} V'$

if $V \subset V' \subset V(x)$

V'/V supported at x

length = dim = i

$n = 1$

$i = 0$ identity $V' = V$

$i = 1$ $V' = V(x) = V \otimes \mathcal{O}_C(x)$

Proof 1: (topol)

$$\pi_1(J(C)) = H_1(J(C)) = H_1(C)$$
$$= \pi_1(C)/[\pi_1, \pi_1]$$

$\pi_1(C)$: $\alpha_1, \dots, \alpha_g$

$\beta_1, \dots, \beta_g$ symp.

$/ \prod_{i=1}^{g} \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1}$

$$\begin{tikzcd} \sigma: \pi_1(C) \arrow[r] \arrow[dr] & Gl_1(\mathbb{C}) \\ & H_1(C) \arrow[u] \end{tikzcd}$$

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proof (2)
(alg geom)

Deligne

$$\mathrm{Sym}^n C \longrightarrow \mathrm{Pic}^n(C)$$
$$(x_1, \ldots, x_n) \longrightarrow \mathcal{O}_C(x_1 + \ldots + x_n)$$

local system descends to $\mathrm{Sym}^n C$, you can show it descends to $\mathrm{Pic}^n(C)$

G	${}^L G$	
GL_n	GL_n	
SL_n	PGL_n	
$SO(2n)$	$SO(2n)$	
$Spin(2n)$	$Pin(2n)$	
$SO(2n+1)$	$Sp(2n)$	$(B_n \Leftrightarrow C_n)$