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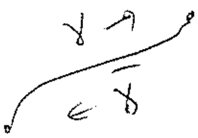
Geometric Flows and Special Holonomy

Holonomy in Riemannian Geometry

(M^n, g) connected Riem. mfld $\longrightarrow$ ∇ Levi-Civita connection



$\gamma : [0,1] \to M$, piecewise C^1

$$P^\nabla_\gamma : T_{\gamma(0)} M \xrightarrow{\sim} T_{\gamma(1)} M$$

$$P^\nabla_{\bar\gamma} = (P^\nabla_\gamma)^{-1} \quad \text{and} \quad P^\nabla_{\gamma_2 \times \gamma_1} = P^\nabla_{\gamma_2} \circ P^\nabla_{\gamma_1}$$

1918 Schouten

$$H_x = \{ P^\nabla_\gamma \mid \gamma(0) = \gamma(1) = x \} \subseteq O(T_x M), \quad \text{subgroup}$$

"degrees of freedom of g"

1925 Cartan

(i) H_x is a Lie group. (Borel-Lichnerowitz)

(ii) If H_x acts reducibly on T_xM then g is locally a product

$$(M, g) \simeq (M_1, g_1) \times (M_2, g_2)$$

(de Rham: globally true if $\pi_1(M) = 0$, g complete)

(iii) When $n = 4$, the subgroup $SU(2)$ occurs as a holonomy group

$$u : T_x M \to \mathbb{R}^n \qquad H_u = u\, H_x\, u^{-1} \subseteq O(n)$$

(iii)(a) $\mathrm{Ric}(g) = 0$

(b) $\mathrm{Riem} = * \mathrm{Riem}$

(c) general solution depends on 2 fcts of 3 variables (modulo diffeo)

Facts about $SU(2)$

$SU(2) \subseteq SO(4)$ $\quad \mathbb{R}^4 \simeq \mathbb{H}. \quad x^0, \ldots, x^3$

Let

$$\Omega_1 = dx^0 \wedge dx^1 + dx^2 \wedge dx^3$$
$$\Omega_2 = dx^0 \wedge dx^2 + dx^3 \wedge dx^1$$
$$\Omega_3 = dx^0 \wedge dx^3 + dx^1 \wedge dx^2$$

$\Omega_1^2 = \Omega_2^2 = \Omega_3^2$

$\Omega_i \wedge \Omega_j = 0, \ i \neq j$

$g \in GL(4, \mathbb{R})$ lies in $SU(2)$ iff $g^*(\Omega_i) = \Omega_i$

$SU(2)$ acts simply trans on $S^3 \subseteq \mathbb{R}^4$

(M^4, g), $u: T_x M \to \mathbb{R}^4$ $\quad H_u = SU(2)$

Then $\exists!$ ∇-parallel 2-forms Υ_i on M s.t.

$$(\Upsilon_i)_x = u^*(\Omega_i)$$

$\nabla \Upsilon_i = 0 \Rightarrow d\Upsilon_i = 0$

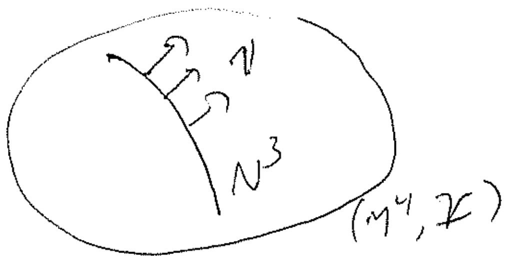
Conversely, if $\Upsilon = \begin{pmatrix} \Upsilon_1 \\ \Upsilon_2 \\ \Upsilon_3 \end{pmatrix}$ is a triple of ~~closed~~ 2-forms

on M^4 satisfying $\Upsilon_i^2 = \Upsilon_j^2 \neq 0 \ \forall i, j$, $\Upsilon_i \wedge \Upsilon_j = 0 \ \forall i \neq j$

and $d\Psi_i = 0$ then $\exists$ metric g on M with holonomy $\subseteq SU(2)$ s.t. Ψ_i are g-parallel.

D.F. problem: classify, up to diffeo, the $SU(2)$-structures (M, Ψ) satisfying $d\Psi = 0$.

<u>Another approach using hypersurfaces</u>



$N^3 \hookrightarrow M^4$ oriented

$\nu : N \to TM$ normal v.f.

$f : N^3 \times (-\varepsilon, \varepsilon) \to M$

$f(p,t) = \exp_{t\nu(p)}(p)$

$$f^*\begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\end{pmatrix} = \begin{pmatrix} dt\wedge\omega_1 + \omega_2\wedge\omega_3 \\ dt\wedge\omega_2 + \omega_3\wedge\omega_1 \\ dt\wedge\omega_3 + \omega_1\wedge\omega_2 \end{pmatrix}$$ where ω's are 1-forms on N (dep on t)

$$g_\omega = \omega_1^2 + \omega_2^2 + \omega_3^2 \qquad *_\omega$$

$$0 = f^*(d\Psi) = 0 \iff \frac{d}{dt}\omega = *_\omega(d\omega) - \frac{1}{2}(*_\omega \wedge d\omega)\,\omega$$
$$\text{and } d\left(*_\omega \omega\big|_{t=0}\right) = 0 \quad (\star)$$

Theorem $\exists\ (N^3, \bar{\omega})$ with $d(*_{\bar\omega}\bar\omega)=0$ that is not C^ω in any local coord. system s.t. ✩ has no solution with $\omega|_{t=0}=\bar\omega$.

In fact, if $(N^3,\bar\omega)$ is not C^ω and satisfies

$$(\dagger) \qquad d(*_{\bar\omega}\bar\omega)=0, \quad *_{\bar\omega}(*_{\bar\omega}\bar\omega\wedge d\bar\omega)=C$$

where C is a constant, then ✩ has no solution with $\bar\omega$ as I.C.

Suppose $(N^3,\bar\omega)$ satisfies $(\dagger)$ and ✩ is solvable. Then the metric $g=dt^2+\omega_1^2+\omega_2^2+\omega_3^2$ has hol. in $SU(2)$. Then g is C^ω in harmonic coords (de Turck-Kazdan). The mean curvature of $N\hookrightarrow(M,g)$ is $H=\frac{1}{2}C$, so it is constant. So N is C^ω in the g-harmonic atlas, so $\bar\omega$ is C^ω in the induced C^ω-structure.

To see that $(\dagger)$ has non-C^ω solution, consider for $x:U\to\mathbb{R}^3$ $(U\subseteq M)$ the $\bar\omega$-harmonic equation

$$\Delta_{\bar\omega}x = d(*_{\bar\omega}dx)+0$$

Now let $U\subseteq\mathbb{R}^3$ be an open set, let $g: U\to GL(3,\mathbb{R})$,

$$\bar\omega = (g(x))^{-1}dx, \quad x: U\hookrightarrow\mathbb{R}^3.$$

$$d(*_{\bar\omega} dx) = 0 \qquad 3$$

$$d(*_{\bar\omega} \bar\omega) = 0 \qquad 3$$

$$*_{\bar\omega}({}^t\omega \wedge d\bar\omega) = C \qquad 1$$

$$7$$

$$D : \Gamma(U, GL(3, \mathbb{R})) \longrightarrow B^7$$

$$\delta_\xi D : \cancel{P_x U} \; gl(3, \mathbb{R}) \to \mathbb{R}^7, \ \xi \neq 0$$

is always surjective.

<u>Chapter II</u> Similar story of $H_u = G_2 \subseteq SO(7)$

$H_u = Spin(7) \subseteq SO(8)$.