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The geometry of Riemannian submersions and foliations

Riemannian submersions

$f: (Q^N, g) \to (M^m, \bar{g})$

f is a Riemannian submersion (R.s.) if $f'(q): T_q Q \to T_{f(q)} M$ is an orthogonal projection onto $T_{f(q)} M$.

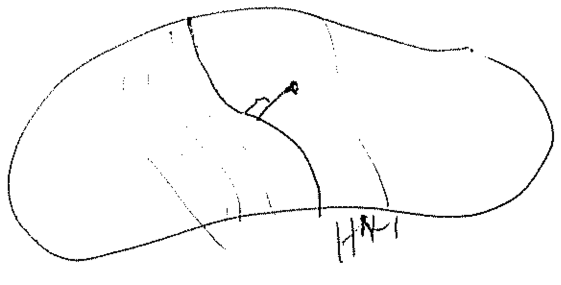
$$\begin{array}{c} Q \\ \downarrow f \\ M \end{array} \qquad f: Q^N \to (M^m, \bar{g})$$

Interesting question: given (Q^N, g), find all such f.

Simple cases

(1) $(m=1)$ $\quad f: (Q, g) \to (\mathbb{R}, dx^2) \quad$ or $(S^1, d\theta^2)$

$\|df\|_g^2 = 1 \quad$ (eikonal equation)



$H \subseteq Q$ oriented hypersurface

$f(q) =$ oriented distance from H

assume $m > 1$ from now on.

$n = N - m.$

$(n,m) = (1,2)$

$$\begin{array}{ccc} (x,y,z) & & (Q^3, g) \\ \downarrow & & \downarrow f \\ (y,z) & & (M^2, \bar{g}) \end{array}$$

locally, $\bar{g} = f(y,z)(dy^2 + dz^2)$

$$g = (dx + f_1(x,y,z)dy + f_2(x,y,z)dz)^2 + f(y,z)(dy^2 + dz^2)$$

$$\bar{g} = f(y,z)(dy^2 + dz^2)$$

(Q^3, g) admitting R.S. of rank 2 depend on 2 fctns of 3 vars up to diffeo but mod diffeo generic g depends on 3 fctns of 3 variables.

Don't always exist.

<u>Remark</u> If (Q^N, g) has a subgroup $H \subseteq Isom(Q^N, g)$ acting on Q in codim m, then $M = H\backslash Q^n$ if a smooth manifold has a unique $\bar{g}$ s.t. $\pi: (Q, g) \to (H\backslash Q, \bar{g})$ is R.S.

<u>Example</u> (Q^3, g) has constant sec'l curvature K, and $H \subseteq Isom(Q^3, g)$ is 1-param subgroup, then $M = H\backslash Q^3$

$SO(3,1)$ $K<0$

$SO(3) \ltimes \mathbb{R}^3$ $K=0$

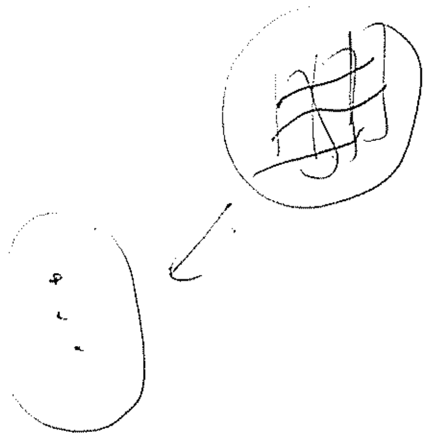
$SO(4)$ $K>0$

Q^3 has const. s.c. K then it has 3-parameter family of tot. geod. surfaces

Let F be a foliation of Q by "planes"

Let $F^\perp$ be the orthogonal foliated by curves.

$M^2 = \mathcal{F}^{\perp} \backslash Q$



with the metric induced by each leaf of $\mathcal{F}$ is a Riemann submersion and it has constant s.c. K

(Type I)

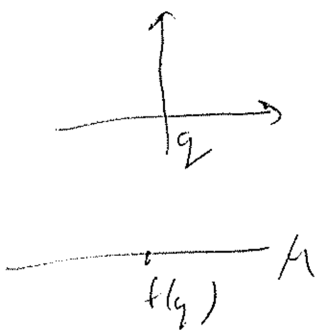
(previous example is Type II)

Prop If (Q^3, g) has const s.c. and $f : (Q, g) \to (M^2, \bar{g})$ is R.S. let $\mathcal{H} = (\ker(f'))^{\perp}$ then

(i) If $\mathcal{H}$ is integrable, f is of type I

(ii) If $\mathcal{H}$ is not integrable, then f is of type II.

$$\begin{array}{cc} g & Q^{n+m} \\ & \downarrow f \\ \bar{g} & M^m \end{array}$$



$$T_q Q = \underset{\displaystyle V_q}{\underset{\shortparallel}{\ker f'}} \oplus \underset{\displaystyle H_q}{\underset{\shortparallel}{(\ker f')^{\perp}}}$$

Two tensors of O'Neill

(1) $\mathrm{II} \in \Gamma(S(V^*) \otimes H)$ second fund form of fibers

(2) $P \in \Gamma(V \otimes \Lambda^2 H^*)$ Integrability tensor of H

<u>Fact</u>: II + P are the only second order info about f.

Type I: $P \equiv 0$ (i.e., H is an integrable plane field)

In this case, H is tangent to a foliation $\mathcal{F}$ ~~by~~ of Q by tot. geodesic m-planes.

Thus, f determines (and is determined by) the family $\Lambda^n \subseteq G_m(Q^{n+m})$ (Λ is the leaves of $\mathcal{F}$).

Conversely, if $\Lambda^n \subseteq G_m(Q^{n+m})$ ~~is~~ satisfies some given conditions, it is the leaves of a foliation $\mathcal{F}$ of $U \subseteq$ ~~$\mathcal{Q}$~~ Q^{n+m}

The condition we need is that the normal n-plane field be integrable. This is expressed as Λ^n being an integral mfld of an ideal $I \subseteq \Omega^*(G_m(Q^{n+m}))$ generated by 2-forms.

<u>Prop</u> When $n=1$, every Λ^1 is an integral. general soln depends on ~~m~~ m functions of 1 variable.

When $n=2$, the ideal I is involutive, genl soln depends on 1 function of 2 variables.

When $n>2$, I is not involutive but $I^{(1)}$ is involutive and genl soln depends on $\binom{n}{2}$ functions of 2 variables.

Type II: When $P \not\equiv 0$

Case $(n,m)=2$

$$\begin{array}{c} Q^4 \\ f \downarrow \\ M^2 \end{array}$$

Prop Suppose that $f: Q^4 \to M^2$ is R.-S. and $H' = H \oplus [H,H]$ has rank 3 everywhere. Then H' is integrable. Moreover, there is a R.-S. $\pi: Q^4 \to Q^3$ and an $h: Q^3 \to M^2$ s.t. $f = h \circ \pi$.

Say that $f: Q^{n+m}_x \to M^n$ is prime if it can't be factored nontrivially as $f = h \circ \pi$, where π is of type I.

Case $(n,m) = (2,3)$

$$\begin{array}{c} Q^5 \\ f \downarrow \\ M^3 \end{array}$$

Prop any such f is either not prime or it is a group quotient.

Prop $(n,m) = (3,2)$ case

$$\begin{array}{c} Q^5 \\ f \downarrow \\ M^2 \end{array}$$

All such prime f are either
(1) group quotients, or
(2) Constructed as integral manifolds of an EDS whose general soln depends on 1 fnct of 2 variables

<u>Conjecture</u> The general R.-S. from a space form (const s.c.K) depends on function of 2-variables at most.