MATHEMATICAL FOUNDATIONS OF ORIENTIFOLDS

UCSB, JULY 18, 2008

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1. INTRODUCTION

THIS TALK IS A PROGRESS REPORT ON AN EXTENSIVE PROJECT WITH

JACQUES DISTLER & DAN FREED

OUR ORIGINAL GOAL WAS TO GIVE A K-THEORETIC FORMULA FOR THE RR CHARGE OF O-PLANES.

WE SUCCEEDED IN A SPECIAL CASE, BUT TO GIVE THE GENERAL FORMULA IS QUITE NONTRIVIAL, AND HAS FORCED US TO UNDERSTAND MORE PRECISELY THE MATHEMATICAL FOUNDATIONS

⊇ THERE IS A LOT OF FORMALISM WHICH WILL BE UNFAMILIAR TO MANY STRING-THEORISTS. SO THIS TALK WILL BE A BROAD OVERVIEW, SKIPPING MANY DETAILS.

A. TWO MOTIVATIONS

α: THERE ARE PROPOSED TYPE II COMPACTIFICATIONS ON $M_4 \times K$ WITH ALL MODULI STABILIZED.

AN ESSENTIAL ELEMENT IN ALL SUCH CONSTRUCTIONS ARE ORIENTIFOLDS (GKP), NEEDED TO EVADE NO-GO THEOREMS (G, MN)

EXTRAORDINARY CLAIMS MERIT EXTRAORDINARY PROOFS: THERE ARE MANY TECHNICAL POINTS WHICH OUGHT TO BE CLEARED UP.

EXAMPLE: SOLVABILITY OF

$$d_H G = (d + H) G = j_{D\text{-brane}} + j_{O\text{-plane}}$$

$\Rightarrow$ $j_D + j_O$ IS TRIVIAL IN $(d+H)$-COHOMOLOGY

* THIS IS NECESSARY, BUT <u>NOT</u> <u>SUFFICIENT</u> FOR TADPOLE CANCELLATION

- THERE ARE EXAMPLES WHERE TADPOLE CONDITIONS ARE SATISFIED AT THE LEVEL OF DIFFERENTIAL FORMS, BUT NOT FOR INTEGRAL COHO.

Moreover, ...

- THERE ARE CONSISTENCY CONDITIONS FOR D-BRANES IN ORIENTIFOLDS WHICH MUST BE STATED AT THE K-THEORY LEVEL.

B: THE MATHEMATICAL FORMULATION IS A VERY TIGHT AND NONTRIVIAL STRUCTURE FORCING US TO UNDERSTAND MORE DEEPLY TWO KEY THEMES

① HOW TO APPLY

TWISTED, EQUIVARIANT, DIFFERENTIAL, GENERALIZED, COHOMOLOGY THEORIES

COMBINED WITH

② FORMULATING

THE THEORY OF A SELF-DUAL FIELD QUANTIZED BY A PONTRYAGIN SELF-DUAL COHOMOLOGY THEORY.

B. SUMMARY OF RESULTS

1. WORLDSHEET FORMULATION OF THE GENERAL ORIENTIFOLD σ-MODEL

2. ORIENTIFOLD SPACETIMES IN A NUTSHELL

3. DEFINITION OF THE RR CHARGE OF O-PLANES @ THE K-THEORETIC LEVEL.

4. COMPUTATION OF THE CHARGE ONCE WE INVERT 2.

5. CONSISTENCY CONDITIONS FOR D-BRANES IN ORIENTIFOLDS

2. THE WORLDSHEET

IT IS NOT TRIVIAL TO FORMULATE CAREFULLY THE ORIENTIFOLDED σ-MODEL (ALL GENUS, INCLUDE FERMIONS...)

FOCUS TODAY ON ONE QUESTION:

WHAT ARE THE TWISTED SECTORS?

REVIEW ORBIFOLDS:

σ MODEL: $\Sigma \xrightarrow{\phi} Y$, RIEMANNIAN

Γ: DISCRETE GROUP OF ISOMETRIES ON Y.

GAUGE Γ-SYMMETRY:

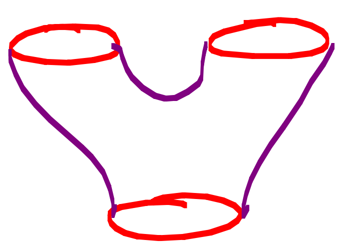
- $\Sigma \longrightarrow$ PRINCIPAL Γ-BUNDLE $\tilde{\Sigma}$
- $\phi \longrightarrow$ EQUIVARIANT MAP $\tilde{\phi}$

$$
\begin{tikzcd}
\tilde{\Sigma} \arrow[r, "\tilde{\phi}"] \arrow[d, "\gamma_{\tilde{\Sigma}}"'] & Y \arrow[d, "\gamma_Y"] \\
\tilde{\Sigma} \arrow[r, "\tilde{\phi}"] & Y
\end{tikzcd}
$$

$\gamma_{\tilde{\Sigma}}$: Fixed point free ; $\Sigma = \tilde{\Sigma}/\Gamma$

(THIS DEFINES A MAP $\phi : \Sigma \to \mathfrak{X}$

$\mathfrak{X} = Y/\!/\Gamma$ A "STACK" .)

$\Sigma =$  IN/OUT BOUNDARIES

- RESTRICTING TO IN/OUT CIRCLE S

  GET A Γ-BUNDLE $S_\Gamma \to S$.

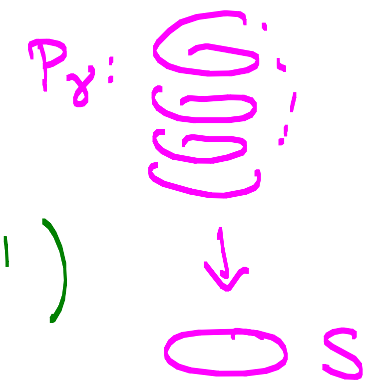
- THESE ARE CLASSIFIED BY HOLONOMY

$$[\gamma] \in \mathrm{Hom}\left(\pi_1(S,*)/\Gamma\right)/\text{Conj.}$$

- WE CAN REPLACE S_Γ BY AN EQUIVALENT BUNDLE

$$P_\gamma = \mathbb{R} \times_{\mathbb{Z}} \langle \gamma \rangle$$

$$(x, \gamma^n) \sim (x+1, \gamma^{n+1})$$



- P_γ IS A SINGLE CIRCLE $\Rightarrow$ SINGLE HILBERT SPACE OF COVERING THEORY

EQUIVARIANT MAP:

$$\begin{tikzcd} P_\gamma \arrow[r, "\phi"] \arrow[d, "\gamma"'] & Y \arrow[d, "\gamma"] \\ P_\gamma \arrow[r, "\phi"] & Y \end{tikzcd}$$

$$\Longrightarrow \quad \boxed{\phi(x+1) = \gamma \cdot \phi(x)}$$

WE RECOGNIZE THESE AS TWISTED SECTORS.

ORIENTIFOLDS: NEW DATA

$$1 \to \Gamma_0 \to \Gamma \xrightarrow{\omega} \mathbb{Z}_2 \to 1$$

$$\Gamma = \Gamma_0 \amalg \Gamma_1$$

AGAIN GAUGE Γ: BUT $\gamma \in \Gamma_1$ IS ACCOMPANIED BY PARITY ON W.S.

$$\begin{tikzcd}
\tilde{\Sigma} \arrow[r, "\tilde{\phi}"] \arrow[d, "\gamma_{\tilde{\Sigma}}"'] & Y \arrow[d, "\gamma_Y"] \\
\tilde{\Sigma} \arrow[r, "\tilde{\phi}"] & Y
\end{tikzcd}$$

- $\gamma_{\tilde{\Sigma}}$ FIXED POINT FREE
- $\tilde{\Sigma}$ ORIENTED SURFACE
- $\gamma_{\tilde{\Sigma}}$ ORIENTATION PRESERVING $\gamma \in \Gamma_0$
- " " REVERSING $\gamma \in \Gamma_1$

AGAIN RESTRICTING TO IN/OUT CIRCLE GET Γ- BUNDLE $S_\Gamma \to S$.

CLAIM: HOLONOMY MUST BE $[\gamma]$ WITH $\gamma \in \Gamma_0$

RECALL WE CAN REDUCE TO P_γ,

BUT P_γ IS TOPOLOGICALLY S^1

$$\boxed{\text{THERE IS NO FIXED-POINT FREE ORIENTATION-REVERSING MAP } S^1 \to S^1}$$

THAT IS WHY THERE ARE NO TWISTED SECTORS ASSOCIATED TO $\gamma \in \Gamma_1$.

NEXT NEED TO MAKE SENSE OF THE B-FIELD AMPLITUDE:

$$\exp\left[2\pi i \int_\Sigma \phi^* B\right].$$

THIS IS QUITE TRICKY: Σ IS UNORIENTED AND IN GENERAL UNORIENTABLE.

- WE NEED THE PROPER MATHEMATICAL HOME FOR B, AND THE MEANING OF $\int_\Sigma$

- EQUIVARIANCE $\Longrightarrow$

$$\boxed{\phi^*(w) = w_1(\Sigma)}$$

- SO IF B IS A W-TWISTED FORM IT CAN BE INTEGRATED

HERE $w \in H^1(\mathfrak{X}; \mathbb{Z}_2)$

IS THE CLASS OF THE DOUBLE-COVER

$$\begin{array}{c} \mathfrak{X}_0 := Y /\!\!/ \Gamma_0 \\ \downarrow \\ \mathfrak{X} := Y /\!\!/ \Gamma \end{array}$$

[RECALL: A DOUBLE-COVER

$$\tilde{M} \longrightarrow M$$

IS A PRINCIPAL $\mathbb{Z}_2$-BUNDLE OVER M.

THEY ARE CLASSIFIED UP TO ISOM.

BY:

$$\mathrm{Hom}(\pi_1(M,*), \mathbb{Z}_2) \cong \mathrm{Hom}(H_1(M), \mathbb{Z}_2)$$

$$\cong H^1(M; \mathbb{Z}_2) \;]$$

$\exists$ MUCH MORE TO SAY...

BUT WE MUST MOVE ON....

3. SPACETIME AS A "STACK"

CATEGORY:

$$\mathcal{X}_1 \underset{p_1}{\overset{p_0}{\rightrightarrows}} \mathcal{X}_0$$

MORPHISMS OBJECTS

Picture: $f \in X_1$

$$a = p_0(f) \xrightarrow{f} b = p_1(f)$$

COMPOSITION:

$$\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, "fg"'] & & c \end{tikzcd}$$

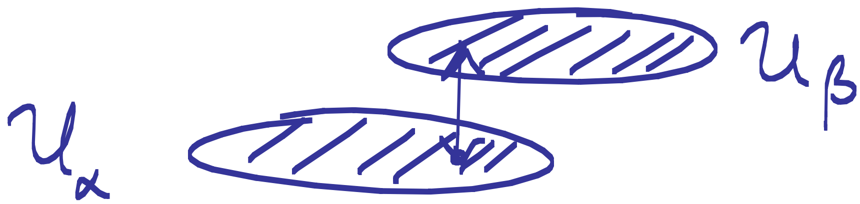
GROUPOID: ALL MORPHISMS INVERTIBLE

VIEW <u>SPACETIME</u> AS A GROUPOID:

$\mathcal{X}_0$, $\mathcal{X}_1$ MANIFOLDS

EXAMPLE A: Y = MANIFOLD WITH OPEN COVER $\{\mathcal{U}_\alpha\}$

$$\mathcal{X}_0 = \coprod_\alpha \mathcal{U}_\alpha \qquad \mathcal{X}_1 = \coprod_{\alpha \le \beta} \mathcal{U}_{\alpha\beta}$$



p_0, p_1 : INCLUSIONS $\mathcal{U}_{\alpha\beta} \rightrightarrows \mathcal{U}_\alpha, \mathcal{U}_\beta$

ISOMORPHISM CLASSES OF OBJECTS IN $\mathcal{X}$

ARE

POINTS OF THE MANIFOLD Y.

BUT THE GROUPOID "REMEMBERS" THE EXTRA DATA OF THE COVERING.

EXAMPLE B: Y = MANIFOLD WITH Γ- ACTION

DEFINES A GROUPOID $\mathcal{X}$ DENOTED $Y/\!/\Gamma$

OBJECTS: $\mathcal{X}_0 = Y$

MORPHISMS: $\mathcal{X}_1 = Y \times \Gamma$

$$\left(y \xrightarrow{\gamma} \gamma \cdot y \right) \begin{array}{l} \xrightarrow{p_0} y \\ \xrightarrow{p_1} \gamma \cdot y \end{array}$$

ISOMORPHISM CLASSES OF OBJECTS IN $Y/\!/\Gamma$
ARE
POINTS OF Y/Γ,

BUT THE GROUPOID "REMEMBERS"
THE STABILIZER GROUP OF POINTS

PUTTING THESE NOTIONS TOGETHER WE GET AN "ORBIFOLD" IN THE SENSE OF SATAKE, BUT TO AVOID CONFUSION WE WILL CALL IT A (DELIGNE–MUMFORD) "STACK."

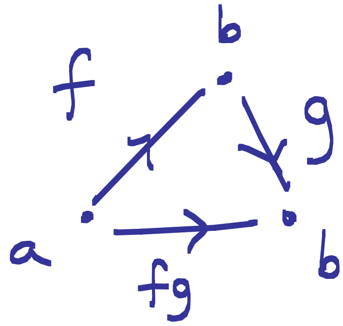
(STACK = EQUIVALENCE CLASS OF GROUPOIDS.)

GIVEN A GROUPOID (OR GENERAL CATEGORY) WE HAVE A SIMPLICIAL SPACE:

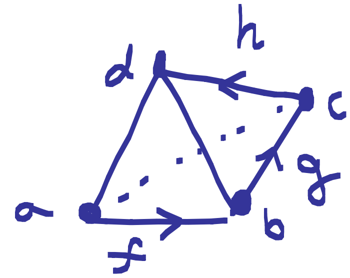
$$\cdots\quad \mathcal{X}_3 \;\substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow}\; \mathcal{X}_2 \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; \mathcal{X}_1 \;\rightrightarrows\; \mathcal{X}_0$$

$\mathcal{X}_n$:= SET OF n-COMPOSABLE MORPHISMS

Point of $\mathcal{X}_2$:



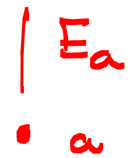
POINT OF $\mathcal{X}_3$:



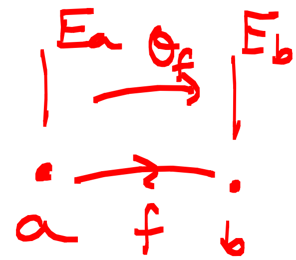
ETC.

DEF: VECTOR BUNDLE ON A GROUPOID $\mathcal{X}$

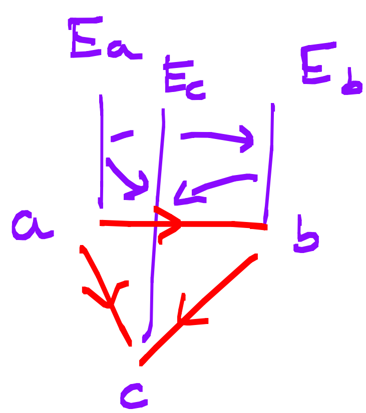
- BUNDLE E OVER $\mathcal{X}_0$



- ISOM. $\theta : p_0^*E \xrightarrow{\simeq} p_1^*E$ OVER $\mathcal{X}_1$



- COMPATIBLE OVER $\mathcal{X}_2$



EXAMPLE A: ORDINARY V.B. ON Y

CONDITION ON X_2: COCYCLE CONDITION

EXAMPLE B: Γ– EQUIVARIANT BUNDLE ON Y.

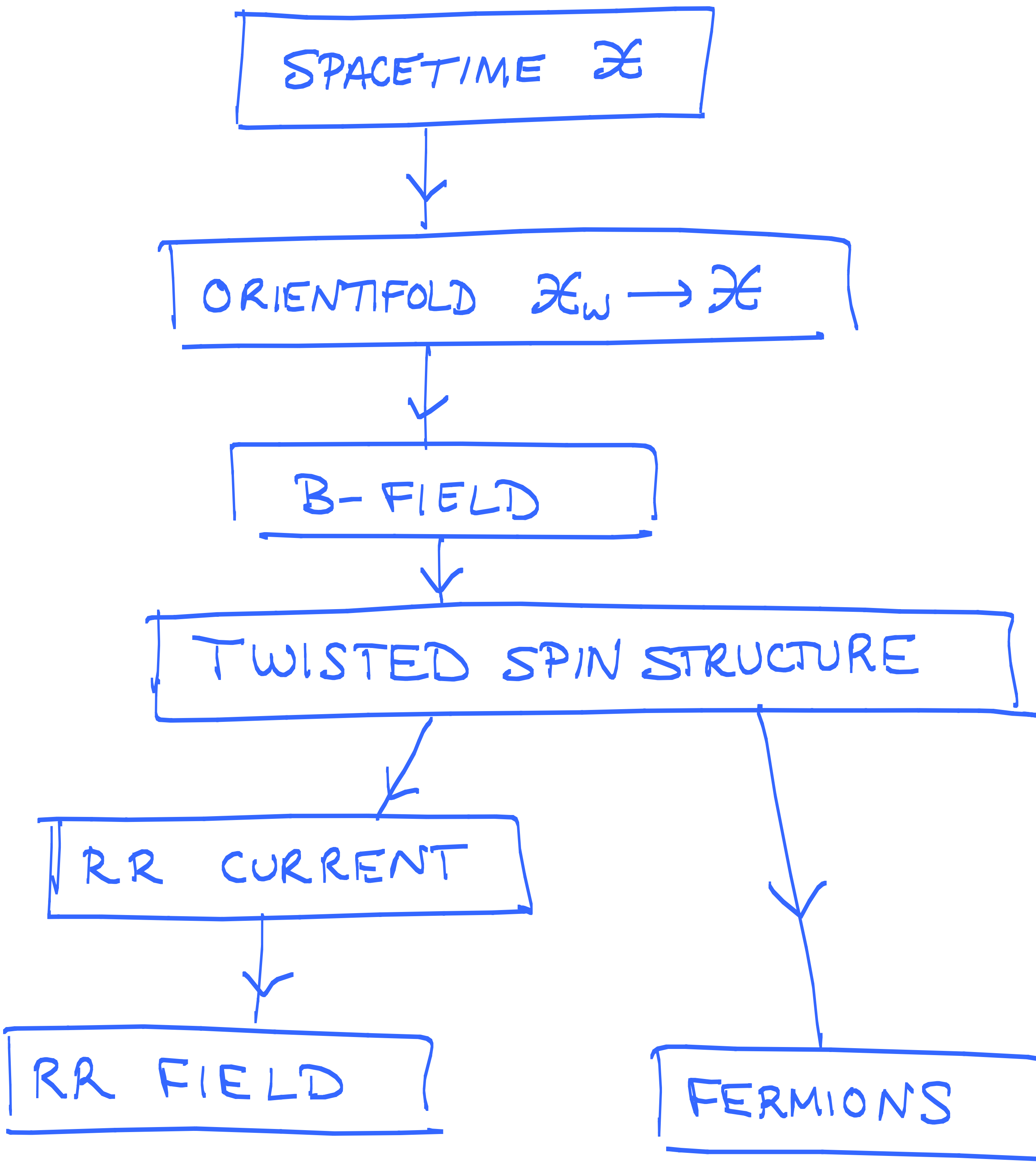
4. ORIENTIFOLD BACKGROUNDS IN A NUTSHELL

[MAIN POINT OF THIS TALK]

WE WANT TO DESCRIBE A TYPE II ORIENTIFOLD SPACETIME IN A CONCISE AND PRECISE WAY.

CLAIM: THE DATA NEEDED TO SPECIFY A PERTURBATIVE, TYPE II, ORIENTIFOLD SPACETIME IS:

WARNING: YOU MIGHT NOT UNDERSTAND MANY WORDS; I WILL SUBSEQUENTLY EXPLAIN (A LITTLE BIT...)



1. <u>SPACETIME</u>: $\mathcal{X}$ IS A SMOOTH D-M STACK EQUIPPED WITH RIEMANNIAN METRIC AND DILATON.

2. ORIENTIFOLD DATA: $\mathcal{X}_w \to \mathcal{X}$ IS A DOUBLE-COVER.

3. B-FIELD: $\check{B}$ IS A GEOMETRICAL TWISTING OF $\check{KR}(\mathcal{X}_w)$.

TOPOLOGICALLY: $[\check{B}] = (t, a, h)$

$$\in H^0(\mathcal{X}, \mathbb{Z}_2) \times H^1(\mathcal{X}, \mathbb{Z}_2) \times H^{3+w}(\mathcal{X}, \mathbb{Z})$$

4. TWISTED SPIN-STRUCTURE

$$w_1(\mathcal{X}) = tw$$

$$w_2(\mathcal{X}) = tw^2 + aw$$

5. RR-CURRENT: ROUGHLY $\check{J}_D \in \check{KR}^{\check{B}}(\mathcal{X}_w)$

6. RR-FIELD: ROUGHLY $d\check{F} = \check{J}_D + \check{J}_0$

$$\Delta\check{F} \in \check{KR}^{-1+\check{B}}(\mathcal{X}_w)$$

5. GENERALIZATIONS OF COHOMLOGY

COHOMOLOGY IS A FUNCTOR

TOP. SPACES $\longrightarrow$ ABELIAN GROUPS

$$X \longrightarrow H^j(X)$$

WE NEED FOUR GENERALIZATIONS:

α. GENERALIZED COHOMOLOGY

β. EQUIVARIANT COHOMOLOGY

γ. TWISTED COHOMOLOGY

δ. DIFFERENTIAL COHOMOLOGY

ε. PUTTING THEM TOGETHER

$$\varepsilon = \alpha + \beta + \gamma + \delta$$

α. GENERALIZED COHOMOLOGY

BASIC PROPERTIES OF NATURALNESS AND MEYER-VIETORIS ALLOW OTHER THEORIES - NOTABLY K-THEORY.

COMMON FEATURE: "SPECTRUM" A SPACE $\mathcal{S}$ SO THAT

$$h(X) = [X, \mathcal{S}]$$

EXAMPLE:

$$H^n(X, \mathbb{Z}) = [X, K(n, \mathbb{Z})]$$

Eilenberg-MacLane spaces

FOR K-THEORY:

$\mathcal{H}$ = $\mathbb{Z}_2$-GRADED HILBERT SPACE

$\mathcal{F}$ = SKEW ADJOINT ODD FREDHOLMS

$$K(X) = [X, \mathcal{F}]$$

PHYSICS: NONCOMMUTATIVE TACHYON

$$T: X \longrightarrow \mathcal{F}$$

HOMOTOPY CLASS OF THE "SOLITON" IS THE K-THEORY CLASS.

$\mathcal{H}$ = "UNIVERSAL CHAN-PATON BUNDLE"

FOR BRANE - $\overline{\text{BRANE}}$ SYSTEMS.

B. EQUIVARIANT COHOMOLOGY

A. EQUIVARIANT COHOMOLOGY

$$H^j_\Gamma(X) := H^j\left(\frac{X \times E\Gamma}{\Gamma}\right)$$

$$= \begin{cases} H^j(X/\Gamma) & \text{FREE} \\ H^j(X \times B\Gamma) & \text{FIXED} \end{cases}$$

$$\boxed{\text{FOR STACKS: } H^j(\mathcal{Y}/\!/\Gamma) := H^j_\Gamma(\mathcal{Y})}$$

- $H^1_\Gamma(Y; \mathbb{Z}_2)$ = DOUBLE COVERS $\tilde{Y} \to Y$ TOGETHER WITH A LIFT OF Γ ACTION TO $\tilde{Y}$
- $\omega_1^{eq}(Y)$: MEASURES WHETHER Γ PRESERVES AN ORIENTATION
- $\omega_2^{eq}(Y)$: MEASURES WHETHER Y ADMITS A SPIN STRUCTURE WITH A SPIN ACTION OF Γ

B. EQUIVARIANT K-THEORY

$\mathcal{H}$ NOW HAS Γ- ACTION

$\mathcal{F}_\Gamma \subset \mathcal{F}$: OPERATORS COMMUTING WITH Γ

$$K_\Gamma(Y) := [Y, \mathcal{F}_\Gamma]$$

i.e. equivariant tachyon field

$$\gamma \cdot T(y) = T(\gamma \cdot y)\,\gamma \qquad \forall \gamma \in \Gamma,\ y \in Y$$

FAMILIAR FROM D-BRANES ON ORBIFOLDS

C. ORIENTIFOLD K-THEORY

$\mathcal{H}$ CARRIES A "REPRESENTATION" OF Γ

Γ_0 : LINEAR

Γ_1 : ANTI-LINEAR

$$ {}^{\omega}K_{\Gamma}(Y) := [Y, \mathcal{F}_{\Gamma}] $$

$$ \gamma T(y) = T(\gamma y) \cdot \gamma $$

- FOR $\Gamma = \mathbb{Z}_2$ & $\omega = Id$

$$ {}^{\omega}K_{\Gamma}(Y) = KR(Y) $$

- FOR STACKS $\mathcal{X} = Y /\!/ \Gamma$

$$ {}^{\omega}K_{\Gamma}(Y) = KR(\mathcal{X}_{\omega}) $$

8. TWISTED COHOMOLOGY

EXAMPLE 1: ANY COHOMOLOGY THEORY $\mathcal{E}$ CAN BE TWISTED BY A DOUBLE COVER

$$\tau : \tilde{X} \to X$$

$\mathcal{E}^{\tau}(X) :=$ ISO. CLASSES OF COCYCLES ON $\tilde{X}$ ODD UNDER τ

EXAMPLE 2: K-THEORY CAN BE TWISTED BY A "GERBE"

REPLACE X BY A GROUPOID WITH X AS THE SET OF ISOMOR. CLASSES

$$X \rightsquigarrow \cdots \mathcal{X}_2 \rightrightarrows \mathcal{X}_1 \rightrightarrows \mathcal{X}_0$$

A (GEOMETRICAL) TWISTING τ OF $K(X)$ IS:

- A ($\mathbb{Z}_2$-GRADED) LINE BUNDLE $L \to \mathcal{X}_1$
- WITH A MULTIPLICATION ON $\mathcal{X}_2$

$$\begin{tikzcd}
 & b \arrow[dr, "L_g"] & \\
a \arrow[ur, "L_f"] \arrow[rr, "L_{gf}"'] & & c
\end{tikzcd}$$

$$\mu : L_f \otimes L_g \xrightarrow{\approx} L_{fg}$$

- AND A COMPATIBILITY ON $\mathcal{X}_3$

<u>EXAMPLE:</u> $X = pt$; $\mathcal{X} = pt/\!/\Gamma$

A TWISTING <u>IS</u> A CENTRAL EXTENSION

$$1 \to \mathbb{C}^* \longrightarrow \widetilde{\Gamma} \longrightarrow \Gamma \to 1$$

- MULTIPLICATION DEFINED BY COCYCLE
- COMPATIBILITY ON $\mathcal{X}_3$: ASSOCIATIVE LAW FOR $\widehat{\Gamma}$

NOW RECALL p. 18 : VECTOR BUNDLES ON GROUPOIDS ...

A TWISTED VECTOR BUNDLE E ON X

WITH TWISTING τ IS

- A VECTOR BUNDLE E ON $\mathcal{X}_0$ (NOT X !!)
- AN ISOMORPHISM $\theta : p_0^* E \otimes L \xrightarrow{\simeq} p_1^* E$ ON $\mathcal{X}_1$

$$\begin{tikzcd} E_a \arrow[r, "\theta_f"] \arrow[d, dash] & E_b \otimes L_f \arrow[d, dash] \\ a \arrow[r, "f"'] & b \end{tikzcd}$$

- COMPATIBILITY ON $\mathcal{X}_2$

EXAMPLE $\quad X = pt \; ; \quad \mathcal{X} = pt /\!/ \Gamma$

A TWISTED V.B. ON X IS A PROJECTIVE REP. OF Γ ;

IN PARTICULAR A REP. OF THE "TWISTING" $\tilde{\Gamma}$.

TWISTINGS OF $KR(\mathcal{X}_w)$

NOW, IN THE MULTIPLICATION

$$L_f \otimes L_g \longrightarrow L_{fg}$$

WE SOMETIMES HAVE COMPLEX CONJ.

$$\overline{L}_f \otimes L_g \longrightarrow L_{fg}$$

FOR EXAMPLE: $X = pt$, $\mathcal{X} = pt//\Gamma$

NOW WE GET A NONCENTRAL EXT.

$$1 \to \mathbb{C}^* \longrightarrow \widetilde{\Gamma} \longrightarrow \Gamma \longrightarrow 1$$

$\Longrightarrow$ FORMULATION OF "DISCRETE TORSION" FOR ORIENTIFOLDS.

8. Differential Cohomology

THE GAUGE INVARIANT INFORMATION IN AN ABELIAN GAUGE FIELD IS

- [MAXWELL] Fieldstrength $F \in \{\text{DIFF FORMS}\}$
- [DIRAC] INTEGRAL COHOMOLOGY CLASS IN SOME GENERALIZED COHOMOLOGY THEORY.
- [AHARONOV-BOHM-WILSON-'tHOOFT] A FLAT FIELD IN SOME COHOMOLOGY WITH $\mathbb{R}/\mathbb{Z}$ COEFF'S.

THESE DATA ARE CORRELATED.

CORRECT CONCEPT IS A DIFFERENTIAL COHOMOLOGY GROUP.

GIVE TWO EXAMPLES

EXAMPLE A: U(1) GAUGE FIELD

$$0 \to \underbrace{H^1(X, \mathbb{R}/\mathbb{Z})}_{\text{FLAT}} \to \check{H}^2(X) \to \underbrace{\Omega^2_{\mathbb{Z}}(X)}_{\text{FIELDSTRENGTH}} \to 0$$

$$0 \to \Omega^1(X)/\Omega^1_{\mathbb{Z}}(X) \to \check{H}^2(X) \to H^2(X, \mathbb{Z}) \to 0$$

TOPOLOGICALLY TRIVIAL GAUGE POTENTIALS MOD GAUGE EQUIVALENCE

TOPOL. CLASS $c_1(L)$

EXAMPLE B: TYPE IIA RR FIELDS IN THE ABSENCE OF RR CURRENT

$$G = G_0 + \bar{u}^1 G_2 + \cdots + \bar{u}^5 G_{10}$$

$$0 \to \underbrace{K^{-1}(X, \mathbb{R}/\mathbb{Z})}_{\text{FLAT RR}} \to \check{K}^0(X) \to \left[\Omega^\bullet_{\mathbb{Z}}(X, \mathbb{R}[u, \bar{u}^1])\right]^0 \to 0$$

$$0 \to \frac{(\Omega^\bullet(X, \mathbb{R}[u,\bar{u}^1]))^{-1}}{\Omega^\bullet_{\mathbb{Z}}(\cdots\cdots)^{-1}} \to \check{K}^0(X) \to \underbrace{K^0(X)}_{\text{TOPOL. CLASS OF RR FIELD}} \to 0$$

$$C = \bar{u}^1 C_1 + \cdots + \bar{u}^5 C_9$$

$$\varepsilon := \alpha + \beta + \gamma + \delta$$

NOW ONE NEEDS TO COMBINE THESE FOUR CONCEPTS.

AN IMPORTANT OUTCOME IS THAT

THE GEOMETRICAL TWISTINGS $\check{B}$ OF $\check{K}R(\mathcal{X}_\omega)$ ARE THEMSELVES TWISTED, DIFFERENTIAL, CLASSES IN A (SMALL) GENERALIZATION OF $\check{H}^3(\mathcal{X})$.

THE TWISTING GENERALIZES THE RULE $B \to -\gamma^*(B)$ $\gamma \in \Gamma$,

TOPOLOGICAL CLASSES OF TWISTINGS

$$\underbrace{H^{3+w}(\mathcal{X};\mathbb{Z})}_{\text{TWISTED BY } \mathcal{X}_w \to \mathcal{X}} \longrightarrow \mathcal{P}^3(\mathcal{X}) \longrightarrow H^0(\mathcal{X},\mathbb{Z}_2)\times H^1(\mathcal{X},\mathbb{Z}_2)$$

$$\text{Top. Class}([\check{B}]) = (t, a, h)$$

$t = 0,1 \iff$ IIA, IIB

$a \in H^1(\mathcal{X};\mathbb{Z}_2) \iff$ WHEN $(-1)^{F_L}$ ACCOMPANIES γ IN W.S. THEORY

$h \in H^{3+h}(\mathcal{X};\mathbb{Z}) \iff$ GENERALIZES FAMILIAR QUANT. LAW OF THE FIELDSTRENGTH

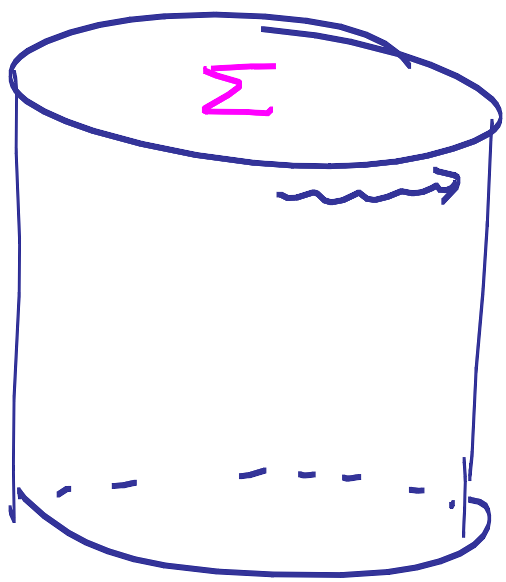
6. SELF-DUAL FIELDS

THE SECOND MAIN INGREDIENT IS THE THEORY OF SELF-DUAL FIELDS APPLIED TO $\check{K}\check{R}^{\check{B}}(\mathcal{X}_{\omega})$.

RECALL THE FAMILIAR PARADIGM

3D ABELIAN CHERN-SIMONS $\Leftrightarrow$ SELF-DUAL SCALAR

$\Sigma \times \mathbb{R}$:



Σ = RIEMANN SURFACE WITH BOUNDARY

CS IN THE BULK IS HOLOGRAPHICALLY DUAL TO CHIRAL EDGE STATES ON THE BOUNDARY.

THIS CONSTRUCTION CAN BE GENERALIZED TO ANY PONTRYAGIN SELF-DUAL COHO. THEORY

FOR EXAMPLE, FOR THE SELF-DUAL FIELD $[\check{F}] \in \check{H}^{2\ell+1}(M_{4\ell+2})$

USE CHERN-SIMONS IN $(4\ell+3)$-DIM'S:

KEY OBSERVATION: $A \in \Omega^{2\ell+1}(M_{4\ell+3})$

$$S(A) = \frac{1}{2}\int_M A\,dA \implies$$

$$S(A_1+A_2) - S(A_1) - S(A_2) = \int_M A_1\,dA_2$$

$$= \int^{\check{H}} \check{A}_1 * \check{A}_2$$

THE CHERN-SIMONS ACTION IS A QUADRATIC REFINEMENT OF THE BILINEAR FORM ON THE DIFF'L COHOMOLOGY!

THE KEY INGREDIENT IN DEFINING A SELF-DUAL THEORY IS THE QUADRATIC REFINEMENT OF A $\mathbb{Z}$ BILINEAR FORM.

WHEN APPLIED TO THE SELF-DUAL RR FIELD IN ORIENTIFOLDS WE NEED AN 11-DIMENSIONAL CHERN-SIMONS THEORY WITH ACTION:

$$\text{``} \quad S(A) = \int^{\check{KO}} A * \bar{A} \qquad A \in \check{KR}(\mathcal{X}_{11}) \quad \text{''}$$

TO MAKE SENSE OF THE ACTION WE NEED THE ISOMORPHISM OF TWISTINGS:

$$\check{B} + \overline{\check{B}} \cong \tau(\check{KO}(\mathcal{X}_\omega))$$

THE EXISTENCE OF THIS ISOM REQUIRES A "TWISTED SPIN STRUCTURE"

$$\boxed{\begin{aligned} w_1(\mathcal{X}) &= tw \\ w_2(\mathcal{X}) &= tw^2 + aw \end{aligned}}$$

A NEW TOPOLOGICAL CONSISTENCY CONDITION ON ORIENTIFOLD BACKGROUNDS

- THE TWISTED SPIN STRUCTURE IS ALSO NEEDED TO FORM THE ACTION FOR GRAVITINOS/DILATINOS

- THE TWISTED SPIN STRUCTURE IS ALSO NEEDED TO CONSTRUCT THE WORLDSHEET THEORY.

7. A FORMULA FOR THE CHARGE OF $Op^{\pm}$ PLANES

PUTTING THESE IDEAS TOGETHER WE GET

A. A DEFINITION OF THE K-THEORETIC CHARGE OF AN O-PLANE

B. A COMPUTATION OF THE CHARGE IN A SPECIAL CASE:

- $\Gamma = \mathbb{Z}_2$
- $B =$ "UNIVERSAL TWISTING" (OFTEN ASSUMED IN LIT.)
- $\mathrm{Cod}(F) = 0 \bmod 4$
- F spin.

$F =$ FIXED POINT LOCUS

$$\boxed{\mu = \mp \frac{1}{2} \imath_* \left(\frac{C(F)}{\psi_{1/2}(\text{EULER}(\nu))} \right) \in KR^0(\mathcal{X}_w)[\tfrac{1}{2}]}$$

- $\imath : F \hookrightarrow Y$ INCLUSION OF FIXED POINT LOCUS, WITH NORMAL BUNDLE ν
- $\text{EULER}(\nu) = \Delta_+(\nu) - \epsilon \Delta_-(\nu)$
- ψ_2 : ADAMS OPERATION

  $\psi_{1/2}$: MULT. INVERSE
- $C(F)$: BOTT'S "CANNIBILISTIC CLASS", A KO-THEORETIC VERSION OF THE WU-CLASS:

$$\int_F^{KO} \psi_2(y) = \int_F^{KO} C(F)\, y \qquad y \in KO(F)$$

PROOF INVOLVES ATIYAH-SEGAL LOCALIZATION THEOREM.

TAKING CHERN CHARACTERS WE REPRODUCE

$$\left[\mu(O_p^{\pm})\right]_{DR} = \pm \, 2^{p-4} \, \nu_* \left(\sqrt{\frac{L(R_T/4)}{L(R_N/4)}} \right)$$

- REPRODUCES MORALES-SCRUCCA-SERONE, SCRUCCA-SERONE

- CONTRADICTS MUKHI-SURYANARAYANA, JULIA-LABORDERE, AND ....
  POSSIBLY .... WITTEN [COMP'T W/OUT VECTOR STRUCTURE] ....

8. CONSISTENCY CONDITIONS FOR D-BRANES

THIS FORMALISM ALSO GIVES CONSISTENCY CONDITIONS ON D-BRANES.

D-BRANE WORLDVOLUME

$$i : W \hookrightarrow Y$$

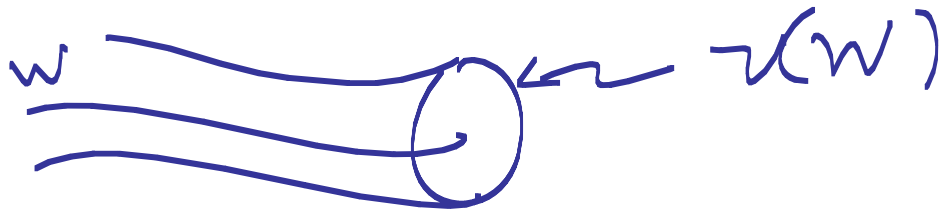
INVARIANT UNDER Γ

GAUGE BUNDLE ON W DEFINES AN ELEMENT $\mathcal{E} \in {}^{\omega}K_{\Gamma}^{\tau_W}(W)$.

TO COMPUTE RR CURRENT $i_*(\mathcal{E})$ NEED

$$i_* : {}^{\omega}K_{\Gamma}^{\tau_W}(W) \longrightarrow {}^{\omega}K_{\Gamma}^{B}(Y)$$

LET $\pi : \nu(W) \longrightarrow W$ BE PROJECTION OF THE NORMAL BDLE



THOM ISOMORPHISM $\Longrightarrow$

$$\boxed{\pi^*(\tau_W + \tau_\nu) = B}$$

$\Longrightarrow$ FAMILIAR RULES FOR WHEN W CARRIES SO OR Sp GAUGE GROUP + MUCH MORE ...

9. CONCLUSIONS

- ∃ LOTS MORE TO SAY ABOUT D-BRANES IN ORIENTIFOLDS & THEIR TWISTED GAUGE THEORIES
- MUCH WORK REMAINS TO BE DONE TO COMPLETE OUR PICTURE
- SOME PUZZLES NEED TO BE RESOLVED
- ∃ FURTHER CONSISTENCY CONDITIONS FOR D-BRANE ANOMALY CANCELLATION
- IT WOULD BE GOOD TO LOOK AT EXAMPLES OF STRING BACKGROUNDS AND EXAMINE THE CONSEQUENCES OF OUR CONSISTENCY CONDITIONS.
- INCLUDE T-DUALITY
- S-DUALITY?
- RELATION TO M & F THEORY?