Morrison: Intoduction to wall crossing formulas ①

Geometry

$M \geq |Z|$

X: CY_3 $\gamma \in H_*(X)$

where are there volume-minimizing cycles in class γ?

$\exists$ calibration on X

$\omega \in \Omega^*(X)$, $d\omega = 0$

$$\int_Y \omega \leq \mathrm{vol}(Y)$$

$Y \in [\gamma]$ $\updownarrow$ BPS charge of the cycles

Mass of D-brane wrapping Y

R. Harvey
B. Lawson ~ 1980?

... ω = Kähler form ~~metric~~

Ω = holom n-form on CY n-fold

Physics

- 4D $\mathcal{N}=2$ SUGRA (type IIA, IIB on CY_3) cpct.
- 4D $\mathcal{N}=2$ SUSY (... on non-cpct. CY_3)

Superalgebra $\supseteq$ Poincaré

reps of superalgebra:

- BPS massive reps

$M \geq |Z|$ BPS charge saturated by BPS states

Q: given a possible "charge" of possible massive states how many BPS states of that charge are there?

A: spectrum of the theory depends on parameters of $\mathcal{N}=2$ theory

$$\sum_{\substack{e \in \text{lattice} \\ \cap \text{ cone}}} n_q \, q^e$$

- gen. for fw. BPS spectrum

3-cycles on CY
calibrated by holom 3-form
"special Lagrangian 3-cycles"
(D3-brane wrapped, in IIB)
} no technol. available

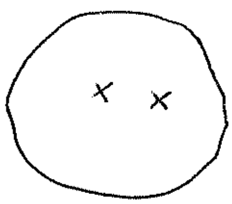
2k-cycles on CY calibrated by ω^k, "complex submflds"
(D2k-brane, in IIA)
} good technology from A.G.

<u>Seiberg-Witten</u>: $\mathcal{N}=2$ susy QFT
SU(2), no matter

$u \in \mathbb{C}$ − bad values (single parameter)

<u>cplx. charge</u>
BPS charge
$Z = Z(u)$
holomorphic

weakly coupled $\longleftrightarrow$ perturbative techn.
⋮
Strongly coupled

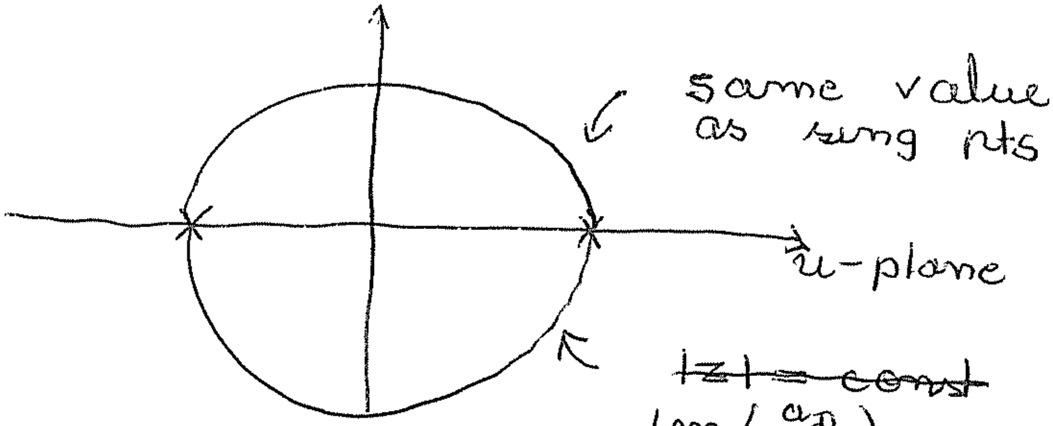


(not single-valued)

spectrum can be calculated at large u

$$\mathcal{M}_{(n_m, n_e)} \neq \emptyset$$
$$\Longleftrightarrow (n_m, n_e) = (0, \pm 1) \text{ or } (\pm 1, k)$$

Monodromy in (n_e, n_m) is $\Gamma(2) \subseteq SL(2;\mathbb{Z})$



~~$|z| = \text{const}$~~

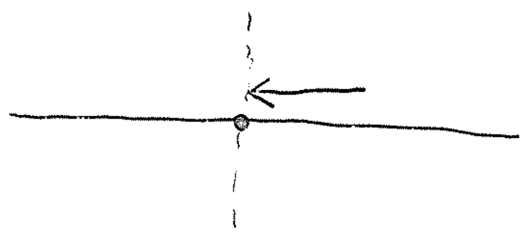
$\mathrm{Im}\left(\frac{a_D}{a}\right) = 0$

$\Leftrightarrow z$ real

- At the 2 bad pts some BPS state becomes massless.

- spectrum can change as you cross ~~at~~ wall
  (BPS spectrum jumping
  $\rightsquigarrow$ jumping wall / jumping line
  (wall of marginal stability)

u - plane $\longleftrightarrow$ cplx Kähler class of CY
$\omega + iB$



stable

"stability" in moduli problems in A.G.

$\left\{ \begin{array}{l} \text{alg. geom objects} \\ \text{labelled by } z_1, \dots z_k \end{array} \right\} \Big/ \begin{array}{l} \text{cplx.} \\ \text{sym. grp} \end{array}$

e.g.

$$(z_1, z_2) \Big/ \mathbb{C}^* \quad (z_1, z_2) \sim (\lambda z_1, \lambda z_2)$$

$(0,0)$ lies in closure of every orbit.

$$\{(z_1, z_2) \mid (z_1, z_2) \neq (0,0)\} \Big/ \mathbb{C}^* \cong \mathbb{CP}^1$$

vector-bundles on Riem. Surf.

$W \subseteq V$

"bad pts" if (or unstable)

$$\frac{\deg(W)}{rk(W)} > \frac{\deg(V)}{rk(V)}$$

<u>Mumford – Takemoto stability</u>

$W \subseteq V$ $\qquad$ X^d proj alg. variety

$H =$ ample divisor

$$\mu_H(W) = \frac{c_1(W) \cdot H^{d-1}}{rk(W)}$$

<u>stable</u>: $\forall \; W \subseteq V$, $rk\, W < rk\, V$

we have $\mu_H(W) < \mu_H(V)$

<u>semistable</u> same w/

$\mu_H(W) \leq \mu_H(V)$

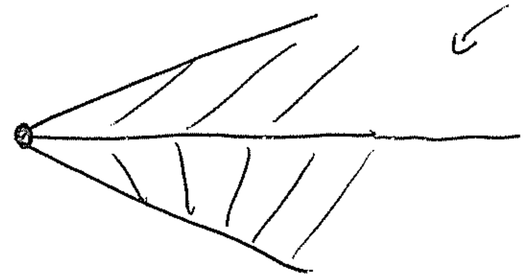
⇝ you have moduli spaces (given H)

$$\mathcal{M}_{s.stab, H} \supseteq \mathcal{M}_{stab, H}$$

cpct.

as you vary H (studied by Thaddeus)

$$H^2(X) \supseteq \mathcal{K}(X) = \text{Kähler cone}$$
$$\cup$$
$$[H]$$



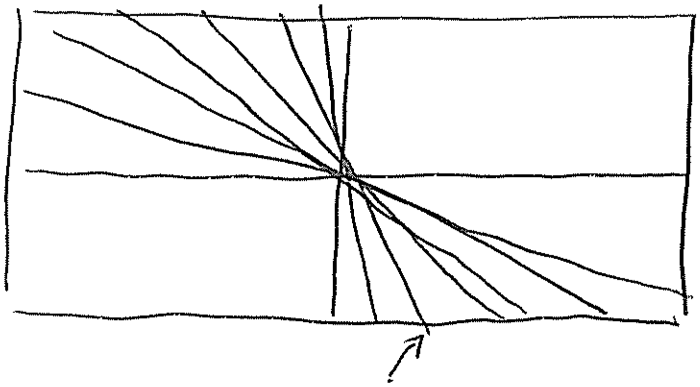
chambers w/ walls

BPS jumping in alg. geom

- formalized by Bridgeland
  " abstract stability conditions!

Nagao – Nakajima

non-cpct. CY



- countable number of lines
- accumulate towards $y = -x$
- Kontsevich – Soibelman

$\rightsquigarrow$ Gaitto – Moore – Neitske