BPS-states in 4d $N=2$ $SU(2)$-theory

$N=2$ susy (twisted) YM-theory w/ gauge group $SU(2)$ on a cpct, simply connected 4-mfld M w/ $2N_f$ hypermultiplets

<u>susy extension</u> of field content:

- E rank 2 cplx vct-bdl, $\downarrow$ A connection, F_A, M — $w_2(E) = w_2(M)$
- A, ϕ — $N=2$ vct-bdl, Higgs, gluinos — $\Gamma(M, S^+ \otimes Ad(E))$
- N_f hypermultiplet $(q^i, \underbrace{\chi^i}_{\text{Weyl spinor}}, \tilde{q}^i, \underbrace{\chi^i}_{\text{Weyl}})$ c.c.

$$F_A^+ = \sum_{i=1}^{N_f} (\chi^i \otimes \bar{\chi}^i)_{00} \qquad i = 1, \dots, N_f$$
$$\not{D}_A \chi^i = 0$$

- $0 \le N_f \le 4$: Beta-fct should be zero
- $u(1)_R$-sym: gluinos charge 1; Weyl — " — 1; Higgs charge 2

Seiberg-Witten moduli space

$\longrightarrow$ Coulomb branch $+$ $\underbrace{\text{Seiberg-Witten}}_{\text{moduli space of soln's to SW-eqns}}$

$$Z_{DW} = \underbrace{Z_u}_{\substack{\text{only contrib} \\ \text{for } b_2^+ = 1}} + Z_{SW}$$

- Coulomb-branch: elliptic fibration over $\mathbb{CP}^1 \ni [u:1]$; rational elliptic surface, birational to $\mathbb{CP}^2$

  fiber: elliptic curve E_u : $y^2 = 4x^3 - g_2(u)x - g_3(u)$

  g_2, g_3: polynomials of deg 2/3

  Torus w/ base pt: (∞, ∞) $\quad \Delta = g_2^3 - 27 g_3^2$

  $\longrightarrow$ non-trivial elliptic fibration (otherwise: $\chi(T \to \underset{\mathbb{CP}^1}{\overset{Z}{\downarrow}}) = \chi(T^2)\chi(\mathbb{CP}^1) = 0$)

  $\longrightarrow$ singular fibers $\Longleftrightarrow$ exactly where some BPS states become massless

  node: $g_2 \neq 0 \neq g_3$ $\quad \Delta = 0$

  cusp: $g_2 = 0 = g_3$ $\quad \Delta = 0$

  $\longrightarrow$ Kodaira classification of singular fibers

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weak coupling limit dictates $I^*_{4-N_f}$

$N_f = 0$	I_4^*	I_1	I_1		
1	I_3^*	I_1	I_1	I_1	I_1
2	I_2^*	I_2	I_2		
3	I_1^*	I_1	I_4		

$N_f = 0$: universal curve over $H/\Gamma_0(4)$, i.e. over τ it's elliptic curve w/ modular parameter τ

$N_f = 2$: related by 2-isogeny $H/\Gamma(2)$

$N_f = 3$: transfer of the (*)

$N_f = 1$: elliptic modular surface (ramified cover of $H/\Gamma_0(6)$)

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we have an analytical marking

$$\Omega = du \wedge \frac{dx}{y}$$

$$\int_{\substack{A\text{-cycle} \\ B\text{-cycle}}} \Omega = \begin{pmatrix} \omega \\ \omega' \end{pmatrix} \otimes du \qquad \boxed{\tau} = \frac{\omega'}{\omega}$$

regular fibers E_u

we can pick symplectic basis of homology

as we go around fibers
they transform by $Sl(2;\mathbb{Z})$

2 holom. sections of a rank 2 vct-bdl

$\omega = \int_{E_u} \eta \wedge \bar{\eta}$

Kähler - metric
special coord
$dp = 4 \operatorname{Re}(\omega\, du)$
$dq = -4 \operatorname{Re}(\omega'\, du)$

$= dp \wedge dq$

$\rightsquigarrow$ special Kähler connection ∇
$\nabla(\partial_p) = \nabla(\partial_q) = 0 \qquad d^{\nabla}(\nabla) = 0$
$\nabla(\omega) = 0$

$\Xi = \omega(\pi^{1,0}, \nabla \pi^{1,0})$ holom. cubic form
$= \left(\frac{\partial \tau}{\partial a}\right) (\omega\, du)^{\otimes 3}$

$\boxed{\Xi, \tau}$ $\rightsquigarrow$ describe the $\mathcal{N}=2$ low energy effective Lagrangian

measures how much ∇ fails to preserve compl str.

$\omega = \frac{da}{du} \qquad \omega' = \frac{da_D}{du}$ locally

section $MW(\pi)$ pos def pairing ⑤
lattice
$\subset Q(E_8)$

for each root α $\rightsquigarrow$ S_α section
at most 240

$$\mathcal{P} = \nabla \lambda_{SW} \in H^1(Z - \coprod_\alpha S_\alpha / \mathcal{U}P)$$

Gauss Manin-connection

λ_{SW} meromorphic 1-form

Thm :
- λ_{SW} exists and is unique up to a holom 1-form
- $m_\alpha := \mathrm{Res}_{P_\alpha} \lambda$ const residues along polar locus

$$[\omega] = -2\pi i \sum_{\alpha \in \Delta} \mathrm{Res}_{P_\alpha} \lambda \cdot [P_\alpha]$$

cplx curve understood as coh

is flat

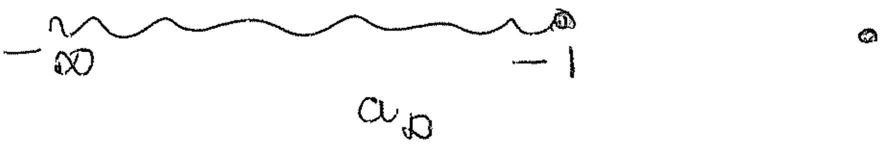
bare masses $= 0$
- for $N_f = 0, 2, 3$ there are no section which are not torsion

$N_f = 1$ $\rightsquigarrow$ one section $\mathrm{Res}_P \lambda = 0$

$$\begin{pmatrix} \omega' \\ \omega \end{pmatrix} \longmapsto M \begin{pmatrix} \omega' \\ \omega \end{pmatrix}$$

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \longmapsto M \begin{pmatrix} a_D \\ a \end{pmatrix}$$ no additive const

a, a_D
are analytical continued



(hypergeom eqn. 3 − regular sing.
~~Heun type 4 − regular sing.~~)

there is a well-defined
line budle $\mathbb{L} \hookrightarrow \mathbb{H}$

• $\nabla \varsigma = \pi^{(1,0)}$ immersion

$$\text{curvature of } L = \Omega$$

• geom quantization

$$Q\big|_{\Lambda \times \Lambda} \subset \mathbb{Z}$$

statements

$$\begin{pmatrix} n_e \\ n_m \end{pmatrix} \triangleq (n_e \ \ n_m)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \partial_p \\ \partial_q \end{pmatrix}$$

$$= n_m \partial_p - n_e \partial_q$$

$$d(\ \ ) = 0 \quad \longleftrightarrow \quad \nabla(\ \ ) = 0$$

go class ✓

$$\begin{pmatrix} n_e \\ n_m \end{pmatrix} \longmapsto \begin{pmatrix} n_e' \\ n_m' \end{pmatrix} = M\begin{pmatrix} n_e \\ n_m \end{pmatrix}$$

$$(n_e' \ \ n_m')\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \partial_p' \\ \partial_q' \end{pmatrix}$$

$$= (n_e \ \ n_m)\, M^{T} \cdot -1\, M \begin{pmatrix} \partial_p \\ \partial_q \end{pmatrix}$$

$$= \text{original}$$

$$\begin{pmatrix} \omega' \\ \omega \end{pmatrix} \longmapsto \omega\,\partial_p - \omega'\,\partial_q = \partial_\omega$$

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \longmapsto S$$

$$d(\ \ ) = \begin{pmatrix} \omega' \\ \omega \end{pmatrix} \otimes du \longmapsto \nabla S = \partial_u \otimes du \ ✓$$

$$Z = Q\left(\begin{pmatrix} n_e \\ n_m \end{pmatrix}, \begin{pmatrix} a_D \\ a \end{pmatrix}\right) = \Omega(p, S)$$

$$= n_e a - n_m a_D$$

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- $u \longmapsto -u$

$$j(u) = \frac{g_2^3}{\Delta} \longmapsto j(-u)$$

$$E_u \cong E_{-u}$$

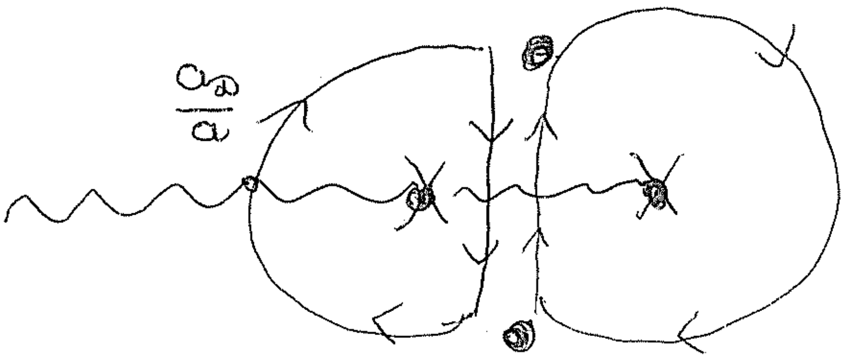
$$\tau \longmapsto \tau + 2$$

$$\begin{pmatrix} \omega' \\ \omega \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix} \begin{pmatrix} \omega' \\ \omega \end{pmatrix}$$

$$M_\infty = \begin{pmatrix} -1 & 4 \\ & -1 \end{pmatrix} \qquad M_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 - n_e n_m & n_e^2 \\ -n_m^2 & 1 + n_e n_m \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 0 \\ \pm \end{pmatrix}$$



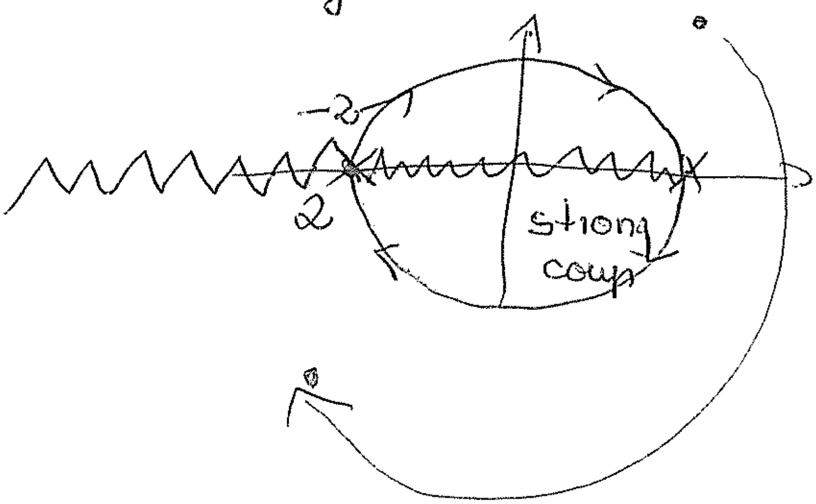
$\mathrm{Im}\, u > 0$: $M'_{-1} M_1 = M_\infty$

$\mathrm{Im}\, u < 0$: $M_1 M_{-1} = M_\infty$

$$M'_{-1} : \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix} \quad \pm \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$M_{-1} : \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix} \quad \pm \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

curve of marginal stability $\quad \frac{a_D}{a} \in \mathbb{R}$



weak coupling

$$\begin{pmatrix}1&2\\0&1\end{pmatrix} \pm \begin{pmatrix}2\\-1\end{pmatrix} = \mp \begin{pmatrix}0\\1\end{pmatrix}$$

$$\begin{pmatrix}1&2\\0&1\end{pmatrix} \pm \begin{pmatrix}0\\1\end{pmatrix} = \pm \begin{pmatrix}2\\1\end{pmatrix}$$

$\rightsquigarrow \pm\begin{pmatrix}2n\\1\end{pmatrix} \quad n \in \mathbb{Z}$ weak coupling

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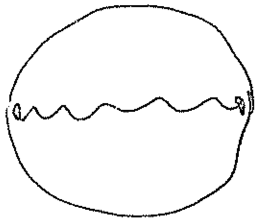
strong coupling



* $M_1 : S_- \to S_+$

- $\begin{pmatrix}1&0\\-1&1\end{pmatrix} \pm \begin{pmatrix}0\\1\end{pmatrix} = \pm\begin{pmatrix}0\\1\end{pmatrix}$
- $\begin{pmatrix}1&0\\-1&1\end{pmatrix} \pm \begin{pmatrix}2\\1\end{pmatrix} = \pm\begin{pmatrix}2\\-1\end{pmatrix}$
- $\begin{pmatrix}1&2\\0&1\end{pmatrix} \pm \begin{pmatrix}0\\1\end{pmatrix} = \pm\begin{pmatrix}2\\1\end{pmatrix}$

$\begin{pmatrix}1&2\\0&1\end{pmatrix} \pm \begin{pmatrix}2\\-1\end{pmatrix} = \mp\begin{pmatrix}0\\1\end{pmatrix}$

$$\begin{pmatrix} n_e \\ n_m \end{pmatrix} \longmapsto \begin{pmatrix} n_e + 2k n_m \\ n_m \end{pmatrix}$$

$n_m \neq 0$: $\frac{n_e}{n_m} + 2k_0 \; [-2, 0]$

$$\left(\frac{a_D}{a}\right) = \frac{n_e}{n_m}$$

becomes massless
but we know all the
massless state $\rightsquigarrow n_m = \pm 1$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\frac{n_e}{n_m} = -1$$

$$\frac{a_D}{a} = -1$$

$\rightsquigarrow$ so there would
be another
massless state

$$\begin{pmatrix} n_e \\ n_m \end{pmatrix} \Leftrightarrow \begin{pmatrix} n_e + 2n_m \\ -n_e - n_m \end{pmatrix} \qquad \frac{n_e}{n_m} = r$$

$$\frac{a_D}{a} = -\frac{n_e + 2n_m}{n_e + n_m} = -\frac{r+2}{r+1}$$

$\leftarrow$ one of these is between $(-2,0)$

$$\begin{pmatrix} 2n \\ 1 \end{pmatrix} \rightarrow n \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} + (n+1) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\frac{a_D}{a}(u^*) = r \quad \in [-2;0]$$

- $$|2na - a_D| = |2n - r|\,|a| = (2n + |r|)\,|a|$$
- $$n\,|2+r|\,|a| = n\,(2-|r|)\,|a|$$
- $$(n+1)\,|r|\,|a| = \underbrace{(n+1)\,|r|\,|a|}_{\text{mass - conversation}}$$

- charge conversation
- mass —— " ——