3D $\mathcal{N} = 4$ Supersymmetric Gauge Theories and Hyperkähler Metrics

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Scalar Theory

Circle-valued scalar $\phi : M^{(3)} \to S^1$ with Lagrangian

$$\mathcal{L} = \frac{\Lambda}{4\pi} \int d\phi \wedge *d\phi$$

"Big Theory"

$$\sum_{\mathcal{L}} \int \mathcal{D}\phi \mathcal{D}A \mathcal{D}B \exp\left[-\frac{\Lambda}{4\pi}\int D_B\phi \wedge *D_B\phi + \frac{i}{2\pi}\int F_A \wedge B\right]$$

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"Dual Theory"

$$\mathcal{L} = \sum_{\mathcal{L}} \int \mathcal{D}A \exp\left[-\frac{1}{4\pi\Lambda} F_A \wedge * F_A\right]$$

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- FI D-terms

- Potential energy

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- $G \to U(1)^r$

$G = SU(2)$

Example

- $$\phi_i = \begin{pmatrix} a_i & 0 \\ 0 & -a_i \end{pmatrix}$$
- Moduli space of vacua: $(\mathbb{R}^3 \times S^1)/\mathbb{Z}_2$
- Classical Metric:

$$ds^2 = \frac{1}{e^2} \sum d\phi_i^2 + e^2 d\sigma^2$$

Quantum Corrections

- Region at infinity in $\mathbb{R}^3$ asymptotically S^2 with radius $|\phi|$
- S^1 with fixed circumference e
- Possibly non-trivial fibration $S^1 \rightarrow S^2$.
- $$ds^2 = \frac{1}{e^2}\sum d\phi^2 + e^2\left(d\sigma - sB_i(\phi)d\phi^i\right)^2$$
- Can be generated at 1-loop.

Low energy effective theory on $\mathbb{R}^3 \times S^1$.

The low enerfy effective theory on $\mathbb{R}^3 \times S^1_R$ should interpolate between $\mathcal{N} = 4$ gauge theory in 3D as $R \to 0$ and $\mathcal{N} = 2$ gauge theory in 4D as $R \to \infty$.

- Massless scalars from holonomy

$$\tilde{\phi}^I = \oint_{S^1} A^I_4 dx^4$$

- "Magnetic Wilson lines"

$$\tilde{\phi}_I = \oint_{S^1} (A_{D,4})_I dx^4$$

obtained from dualizing the 3D gauge field

$R \to \infty$ Limit

Want to study gauge theory at energy scale μ where

- $\mu << \Lambda$
- $\mu << 1/R$

The $R \to \infty$ limit lets us read off spectrum of 4D gauge theory.

3D Moduli space is a $2r$ dimensional torus fibration over the 4D vector multiplet moduli space.

$$\mathcal{J} \rightarrow \mathcal{M}_v$$

where $\mathcal{J}$ is parametrized by the electric and "magnetic" Wilson lines $(\tilde{\phi}^I, \phi_I)$ Denote the fiber over a point $u \in \mathcal{M}_v$ by $\tilde{\mathcal{J}}_u$.

We take the $R \to \infty$ limit by truncating fields to be independent of x^4. This reduction of the 4D Lagrangian yields

$$\begin{aligned}\mathcal{L}^{(3)} = (\Im\tau)\left(-\frac{R}{2}|da|^2 - \frac{R}{2}F^{(3)} \wedge \star F^{(3)} - \frac{1}{8\pi^2 R}d\tilde{\phi}^2\right) + \\ (\Re\tau)\left(\frac{1}{2\pi}d\tilde{\phi} \wedge F^{(3)}\right)\end{aligned} \tag{1}$$

Dualizing the 3D gauge field A^I to a scalar yields

$$\mathcal{L}^3_{dual} = -\frac{R}{2}(\Im\tau)|da|^2 - \frac{1}{8\pi^2 R}(\Im\tau)^{-1}|d\phi - \tau d\tilde{\phi}|^2$$

Semiflat metric

$$g^{sf} = R(\Im\tau)|da|^2 + \frac{1}{4\pi^2 R}(\Im\tau)^{-1}|dz|^2$$

where

$$dz_I = d\phi_I - \tau_{IJ}d\tilde{\phi}^J$$

Why semiflat?

Recall the semiflat metric

$$g^{sf} = R(\Im\tau)|da|^2 + \frac{1}{4\pi^2 R}(\Im\tau)^{-1}|dz|^2$$

here the torus fibers $\mathcal{J}_u$ are flat. The fibers have volume

$$vol(\mathcal{J}_u) \propto \left(\frac{1}{R}\right)^r$$

In the $R \to \infty$ limit the torus is small.

- What is new physics going from $\mathbb{R}^4 \to \mathbb{R}^3 \times S^1$?
- Still have localized 4D instantons.
- New kinda of instanton: A magnetic monopole or dyon that wraps around S^1_R.
- For large R the action is $I = 2\pi RM$.

How do these new instantons affect the moduli space?
$M = |a_D + na|$ is *not* holomorphic so it cannot correct the distinguished complex structure on $\mathcal{M}$. However the monopoles do contribute to the metric on $\mathcal{M}$.

Definition

A *Kähler* manifold is a Riemanian manifold X with an almost complex structure J which is covariantly constant.

Definition

A *hyperkähler* manifold is a Riemanian manifold (M, g) with an three covariantly constant almost complex structures I, J, K which satisfy

$$I^2 = J^2 = K^2 = -1 \qquad IJ = -K, JK = -I, KI = -J$$

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- $K3$ surfaces

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- Moduli spaces from physics!

Definition

The *twistor space* Z of a hyperkähler manifold M is the product

$$Z = M \times S^2$$

equipped with the almost complex structure

$$\underline{I} = (aI + bJ + cK, I_0)$$

- Topologically Z is a trivial bundle.

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Example

Holomorphic line bundles on an elliptic curve

$$Pic\ X \cong Jac\ X \times \mathbb{Z}$$

Complex Structure of Z

- What are the (1,0) forms for complex structure $\underline{I}$?
- Suppose ϕ is a $(1,0)$ form on M in complex structure I.

$$I\phi = i\phi$$

- Define $\theta = \phi + \zeta K\phi$
- Quick calculation:

$$\underline{I}\theta = i\theta.$$

Let $\phi_1, \ldots, \phi_n$ be a basis of $(1,0)$ forms for (M, I).
Then $\phi_i + \zeta K \phi_i$ and $d\zeta$ are a basis of (1,0) forms for $(Z, \underline{I})$.

- The holomorphic (2,0) form

$$\omega_+ = \omega_2 + i\omega_3$$

- can locally be written as

$$\omega_+ = \sum \phi_i \wedge \phi_{n+i}$$

- Define

$$\frac{1}{2}\overline{\omega} = \sum \hat{\phi}_i \wedge \hat{\phi}_{n+i}$$

- Calculation

$$\overline{\omega} = \omega_+ + 2\zeta\omega_1 - \zeta^2\omega_-$$

Theorem (HKLM)

Let M^{4n} be a hyperkähler manifold and Z its twistor space. Then

(i) *Z is a holomorphic fiber bundle $p : Z \to \mathbb{CP}^1$ over the projective line.*

(ii) *the bundle admits a family of holomorphic sections each with normal bundle isomorphic to $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$.*

(iii) *there exists a holomorphic section $\overline{\omega}$ of $\Lambda^2 T_F^* \otimes \mathcal{O}(2)$ defining a sympletic form on each fiber*

(iv) *Z has a real structure τ compatible with (i),(ii),(iii) and inducing the antipodal map on $\mathbb{CP}^1$.*