$\mathcal{N} = 2$ supersymmetric gauge theory and Mock theta functions

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(joint work with Ken Ono)

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q-series in differential topology

"Theorem" (M-Ono)
The following q-series is a **topological invariant**

$$M(q^8) := q^{-1} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} q^{8(n+1)^2} \prod_{k=1}^{n} (1 - q^{16k-8})}{\prod_{k=1}^{n+1} (1 + q^{16k-8})^2}$$
$$= -q^7 + 2q^{15} - 3q^{23} + \cdots .$$

Remark

This series appears in Ramanujan's "Lost Notebook".

Overview:

1. $SU(2)$ and $SO(3)$ gauge theory in mathematics/physics
2. The Coulomb branch and the u-plane integral
3. $SO(3)$-Donaldson invariants for CP^2 and Mock theta functions

Setup

- (X, g): four-dim., compact, smooth, simply connected manifold w/ Riemannian metric g,
  Hodge star $* : \Lambda^p(X) \to \Lambda^{4-p}(X)$ with $*^2 = (-1)^p$;
- Simplest example of a manifold which is *not* of simple type:
  $X = CP^2$ w/ the Fubini-Study metric g,
  g is Kähler, has positive scalar curvature,
  self-dual Kähler form $\omega \in H^2(X) = H^{2+}(X)$,
  $\check{H} = \mathrm{PD}(\omega) \in H_2(X)$;
- ω determines homology orientation:
  any element $\Sigma \in H_2(X)$ is given by $S = \int_\Sigma \omega \in \mathbb{Z}$;

$SU(2)$ gauge theory

- $P \to X$ is $SU(2)$-principal bundles, classified by second Chern class $c_2(P)[X] = k$;
- $\mathcal{A} \in \mathfrak{A}$ connection on P, $\mathfrak{A}^*$ the subset of irreducible connections, gauge group $\mathfrak{G}$;
- $c_2(P)[X] = \frac{1}{8\pi^2} \int_X \mathrm{tr}(F_\mathcal{A} \wedge F_\mathcal{A}) = \frac{1}{8\pi^2} \left(||F_\mathcal{A}^-||^2 - ||F_\mathcal{A}^+||^2 \right)$;
- anti-selfdual instanton = connection $\mathcal{A} \in \mathfrak{A}^*$ w/ $*F_\mathcal{A} = -F_\mathcal{A}$;
- moduli space of solutions $\mathfrak{M}_k \subset \mathfrak{A}^*/\mathfrak{G}$;
  $\mathfrak{M}_k$ is a smooth and oriented manifold of dimension $2d_k = 8k - 3(1 + b_2^+) = 8k - 6$;

Universal bundle construction

- the universal bundle is the $SU(2)$-principal bundle:

$$\mathfrak{L} = (P \times \mathfrak{A}^*)/\mathfrak{G} \to X \times \mathfrak{A}^*/\mathfrak{G};$$

- $\mathfrak{L}$ has a natural connection $\mathfrak{D}$ with curvature $\mathfrak{F} \in \Lambda^2(X \times \mathfrak{M}) \otimes \mathfrak{su}(2)$;

$$p_1(\mathfrak{L}) = -\frac{1}{8\pi^2}\mathrm{tr}(\mathfrak{F} \wedge \mathfrak{F}) = \sum_i \beta_i \otimes \gamma_i, \qquad \begin{array}{ccc} \beta_i & \in & H^*(X, \mathbb{Q}) \\ \gamma_i & \in & H^*(\mathfrak{M}, \mathbb{Q}) \end{array},$$

$$\mu(\alpha) = p_1(\mathfrak{L})/\alpha = \left(\int_\alpha \beta_i\right) \gamma_i \ , \qquad \alpha \in H_*(X) \ ;$$

- for $X = CP^2$, two interesting classes: $p \in H_0(X)$, $\check{H} \in H_2(X)$: $\mu(p) \in H^4(\mathfrak{M})$, $\mu(\check{H}) \in H^2(\mathfrak{M})$;

Donaldson invariants

- Donaldson invariants are the linear function:

$$\mathbf{\Phi}_k : \mathrm{Sym}_* \Big(H_0(X;\mathbb{Z}) \oplus H_2(X;\mathbb{Z}) \Big) \to \mathbb{Q}$$

$$\mathbf{\Phi}_k(p^m, \Sigma^n) = \int_{\mathfrak{M}_k} \mu(p)^m \mu(\Sigma)^n \ ;$$

- invariants can be assembled in a generating function

$$X = CP^2 : \quad \mathbf{Z} = \sum_{k=1}^{\infty} \sum_{m,n=0}^{\infty} \frac{p^m}{m!} \frac{S^n}{n!} \mathbf{\Phi}_k \left(p^m, \check{H}^n \right) \ ;$$

Results for $X = CP^2$

Ellingsrud and Göttsche computed (hard work) the Donaldson invariants for CP^2 of degree smaller or equal to 50:

Theorem (Ellingsrud and Göttsche, 1998)

(k, d_k)	$\mathbf{Z}(p, S) =$
$(1,1)$	$-\frac{3}{2}S$
$(2,5)$	$S^5 - p\,S^3 - \frac{13}{8}p^2\,S$
$(3,9)$	$3\,S^9 + \frac{15}{4}p\,S^7 - \frac{11}{16}p^2\,S^5 - \frac{141}{64}p^3\,S^3 - \frac{879}{256}p^4\,S$
$(4,13)$	$54\,S^{13} + 24\,pS^{11} + \frac{159}{8}p^2S^9 + \frac{51}{16}p^3S^7 - \frac{459}{128}p^4S^5 - \frac{1515}{256}p^5S^3 - \frac{36675}{4096}p^6S$
$\vdots$	$\vdots$

$SO(3)$ gauge theory

- $V \to X$ is $SO(3)$-principal bundles, classified by first Pontryagin class $p_1(V)[X] = -l$ and $w_2(V)$;
- $p_1(V)[X] = -\frac{1}{8\pi^2}\int_X \mathrm{tr}(F_{\mathcal{A}} \wedge F_{\mathcal{A}})$,
- moduli space of asd instantons $\widetilde{\mathfrak{M}}_l \subset \mathfrak{A}^*/\mathfrak{G}$; $\widetilde{\mathfrak{M}}_l$ is a smooth and oriented manifold of dimension $2d_l = 2l - 3(1 + b_2^+)$;
- if $w_2(V) = 0$ then $V = \mathrm{Ad}(P) = P \times_{SU(2)} \mathfrak{so}(3)$ and $p_1(\mathrm{Ad}P) = -4c_2(P)$ (from trace identity);
- for moduli spaces which do not arise from $SU(2)$-bundles take $p_1(V)[X] \equiv w_2(V)^2[X] \mod 4$, for $X = CP^2$: $w_2^2(V)[X] \equiv 1 \mod 4 \Rightarrow l = 3, 7, 11, \ldots$

Reducible connections

- V is a reducible: $V = L \oplus \epsilon$ w/ $F_{\mathcal{A}} = F_A \otimes i\sigma^3$,
  ϵ: trivial oriented real line bundle,
  L: line bundle w/ $c_1(L) = \frac{1}{2\pi} F_A$:

  $$p_1(V)[X] = -\frac{1}{8\pi^2}\int_X \mathrm{tr}(F_{\mathcal{A}} \wedge F_{\mathcal{A}}) = \frac{1}{4\pi^2}\int_X F_A \wedge F_A = c_1^2(L)$$

  for stable classes: $w_2(V) = w_2(L) \equiv c_1(L) \mod 2$
  $\Rightarrow p_1(V)[X] \equiv w_2^2(V)[X] \mod 4$;
- for $X = CP^2$:

  $$\begin{aligned} w_2(V) &\equiv 0 \mod 2: & c_1(L) &= 2n\,\omega \\ w_2(V) &\equiv \omega \mod 2: & c_1(L) &= (2n+1)\omega \end{aligned}$$

Results for $X = CP^2$

Theorem (Kotschick and Lisca, 1995)

(l, d_l)	$\mathbf{Z}_{w_2=1}(p, S) =$
$(3, 0)$	-1
$(7, 4)$	$-3\,S^4 - 5\,p\,S^2 - 19\,p^2$
$(11, 8)$	$-232\,S^8 - 152\,p\,S^6 - 136\,p^2\,S^4 - 184\,p^3\,S^2 - 680\,p^4$
$(15, 12)$	$-69525\,S^{12} - 26907\,p\,S^{10} - 12853\,p^2\,S^8 - 7803\,p^3\,S^6$ $-6357\,p^4\,S^4 - 8155\,p^5\,S^2 - 29557\,p^6$
$\vdots$	$\vdots$

Theorem (Zagier and Göttsche, 1998)

Explicit formula for all coefficients in terms of Jacobi ϑ-functions.

Physics interpretation: Witten, 1988

- $\exists$ twisted $\mathcal{N} = 2$ supersymmetric topological $SU(2)$ or $SO(3)$-Yang-Mills theory on M;
- bosonic fields = differential forms, values in $\mathrm{Ad}(P) = P \times_G \mathfrak{g}$; $\mathcal{A}$ connection on $V \to X$, $\Phi \in \Gamma(X, \mathrm{Ad}(P))$;
- twisted = supersymmetry charge $\mathcal{Q}$ w/ $\mathcal{Q}^2 = 0$ is a scalar, fermionic BRST-operator;
  $\mathcal{Q}$: exterior derivative on $X \times \mathfrak{A}^*/\mathfrak{G} \supset X \times \mathfrak{M}$;
- action: $S = p_1(V) + \{\mathcal{Q}, \dots\}$;
- observables = $\mathcal{Q}$-cohomology classes of $H^*(\mathfrak{M})$ (hence topological invariants)
- expectation values = cup-product to the top-degree evaluated on fundamental class;

Physics interpretation: Seiberg, Witten, 1994

- moduli space of vacua of TQFT

$$\begin{array}{rcccl} & = & \text{Coulomb branch} & + & \text{Seiberg-Witten branch} \\ \mathbf{Z}(p,S) & = & \mathbf{Z}_u(p,S) & + & \mathbf{Z}_{SW}(p,S) \end{array}$$

- Coulomb branch, SW branch: moduli spaces of simpler physical theories;
- Seiberg-Witten branch: moduli space of solutions to (mathematical) SW-equations / gauge transformations;
- for CP^2 or CP^2 with a small number of points blown up: positive scalar curvature + maximum's principle $\Rightarrow$ SW-invariants vanish, $\mathbf{Z}_{SW}(p,S) = 0$;

Low energy effective field theory

- vev $\langle \mathrm{tr}\,\Phi^2\rangle = 2u$ breaks $SU(2)$ or $SO(3) \to U(1)$;
- Coulomb branch = moduli space of vacua of a $\mathcal{N}=2$ supersymmetric $U(1)$-gauge theory;
- classical bosonic fields: connection A on a line bundle $L \to X$ and a complex scalar field φ;
- discrete modulus: $\langle c_1(L)\cup\omega,[X]\rangle = 2k + w_2(V)$, continuous modulus: **a** the minimum of the scalar field, complex gauge coupling: $\tau \in H/\Gamma_0(4)$, $\Gamma_0(4)$ duality group, duality transformation, e.g., $\tau \mapsto \tau+4$;
- bosonic action:

$$S_{\mathrm{bos}} = \int_X \underbrace{\frac{i\,\mathrm{Re}\tau}{16\pi} F_A\wedge F_A}_{=\frac{\pi i\,\mathrm{Re}\tau}{4}c_1^2(L)} + i\, w_2(X)\wedge F_A + \frac{\mathrm{Im}\tau}{16\pi} F_A\wedge * F_A + \mathrm{Im}\tau\,\langle d\varphi, d\bar\varphi\rangle$$

Semi-classical approximation

- path integral for a supersymmetric action S can be defined with mathematical rigor by the stationary phase approx.;
- quadratic approximation $S^{(2)}$ around a critical point is Hessian; $S^{(2)}$ determines a free field theory in the collected variations of the Bose fields $\widetilde{\Phi}$ and Fermi fields $\widetilde{\Psi}$ (= coordinates of the normal bundle at the critical points):

$$S^{(2)} = \int_X \mathrm{vol}_M \quad \left(\left\langle \widetilde{\Phi}, \Delta_{(k,\mathbf{a})} \widetilde{\Phi} \right\rangle + \left\langle \widetilde{\Psi}, i \not{D}_{(k,\mathbf{a})} \widetilde{\Psi} \right\rangle \right) ,$$

where Δ is a family of second-order, elliptic operators and D a family of skew-symmetric first-order operator;

Semi-classical path integral

- functional integration over the fluctuations is infinite-dimensional Gaussian integral, define

$$\int \left[\mathcal{D}\widetilde{\Phi}\ \mathcal{D}\widetilde{\Psi}\right]\ e^{-S^{(2)}} \quad \text{to be} \quad \frac{\operatorname{pfaff} \not{D}_{k,\mathbf{a}}}{\sqrt{\det \Delta_{k,\mathbf{a}}}}\ ;$$

- to integrate this section over the moduli space, check:
  1) line bundle is flat = vanishing of the local anomaly;
  2) no monodromy = vanishing of the global anomaly;
  3) line bundle has canonical trivialization = ratio of determinants is function on the moduli space;

Semi-classical path integral

- **integrate** over continuous moduli, **sum** over the discrete moduli to obtain the semi-classical approximation of the partition function:

$$\mathbf{Z}_u = \sum_k \int d^2\mathbf{a} \quad e^{-S^{(0)}(k,\mathbf{a})} \int \left[\mathcal{D}\widetilde{\Phi}\ \mathcal{D}\widetilde{\Psi}\right] \ e^{-S^{(2)}}$$

$$= \sum_k \int d^2\mathbf{a} \quad e^{-S^{(0)}(k,\mathbf{a})} \frac{\mathrm{pfaff}\, \not{D}_{k,\mathbf{a}}}{\sqrt{\det \Delta_{k,\mathbf{a}}}}$$

- physical considerations guarantee that the semi-classical approximation is exact;

Semi-classical generating function

- we are not only interested in the partition function, but the generating function with the inclusion of observables:

Donaldson theory	$\longrightarrow$	Low Energy Effective Field Theory
$\mu(p)$	$\mapsto$	$2u = \langle \mathrm{tr}\, \Phi^2 \rangle$
$\mu(\Sigma)$	$\mapsto$	$\widehat{T}(u)$

- path integral:

$$\mathbf{Z}_u(p, S) = \sum_{2\,H^2(M;\mathbb{Z})+w_2(V)} \int d\mathbf{a} \wedge d\bar{\mathbf{a}}\; e^{2\,p\,u + S^2\,\widehat{T}(u)}\; e^{-S^{(0)}}\; \frac{\mathrm{pfaff}\, \not{D}_{k,\mathbf{a}}}{\sqrt{\det \Delta_{k,\mathbf{a}}}}$$

Coulomb branch

- Coulomb branch: rational Weierstrass elliptic surfaces, holomorphic fibration $\pi : Z \to CP^1$ where $[u : 1] \in CP^1$:

$$E_u : y^2 = 4\,x^3 - g_2(u)\,x - g_3(u)$$

- Discriminant: $\Delta = g_2^3 - 27 g_3^2$ (smoothness cond. $\neq 0$);
- Kodaira (1950) classified singular fibers: they only depend on the vanishing order of g_2, g_3, Δ;

$u = \pm 1$	node	I_1	0	0	1
	monopole/dyon becomes massless				
$u = \infty$	9 lines meeting in $\widehat{D_8}$	I_4^*	2	3	10
	weak coupling limit				

The u-plane

- analytical marking $du \wedge \frac{dx}{y}$ of elliptic surface:
  period integrals: $\int_{\text{A-cycle}} du \wedge \frac{dx}{y} = \omega du$;
  period integrals: $\int_{\text{A-cycle}} \lambda_{SW} = \mathbf{a}$, $\frac{d\mathbf{a}}{du} = \omega$;
  elliptic fiber E_u is $\mathbb{C}/\langle \omega, \tau\omega \rangle$;
- effective gauge coupling depends holomorphically on scalar component of $\mathcal{N} = 2$ vector multiplet, $\tau = \tau(\mathbf{a})$;
- Oguiso, Shidoa, 1991 classified the Mordell-Weil groups of all rational elliptic surfaces; rational elliptic surface is $\#64$ and universal curve for modular group $\Gamma_0(4)$;

The u-plane

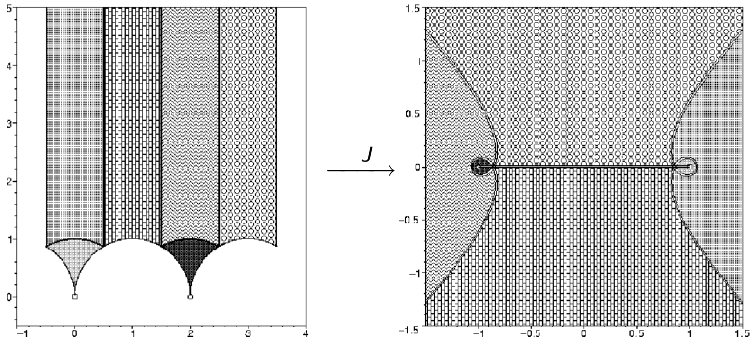

Figure: The mapping from $H/\Gamma_0(4)$ (with the six copies of the fundamental domain) to the u-plane (with the points $u = 1$ ($\square$) and $u = -1$ ($\bigcirc$) removed)

Photon partition function

For $X = CP^2$ we have

$$w_2(X) \equiv \omega \ , \qquad w_2(V) \equiv w_2\,\omega \ ,$$
$$c_1(L) = \frac{1}{2\pi}\,F_A = (2k + w_2)\,\omega \ ,$$
$$S^{(0)} = \int_X \frac{i\,\mathrm{Re}\tau}{16\pi} F_A \wedge F_A + i\,w_2(X) \wedge F_A + \frac{\mathrm{Im}\tau}{16\pi} F_A \wedge *F_A \ ,$$

and

$$\sum_{k\in\mathbb{Z}} e^{-S^{(0)}(k,\mathbf{a})}\,\frac{\mathrm{pfaff}\,\not{D}_{k,\mathbf{a}}}{\sqrt{\det\Delta_{k,\mathbf{a}}}}$$
$$= C \sum_{k\in\mathbb{Z}} e^{-S^{(0)}(k,\mathbf{a})}\,\left[\int_{\check{H}} \left(F_A^+ + \frac{S}{\mathrm{Im}\tau\,\boldsymbol{\omega}}\,\omega\right)\right]$$

Photon partition function

we obtain

$$\sum_{k\in\mathbb{Z}} e^{-i\pi\bar{\tau}(2k+w_2)^2+i\pi\left(k+\frac{w_2}{2}\right)} \left[\left(k+\frac{w_2}{2}\right)+\frac{S}{\mathrm{Im}\tau\,\boldsymbol{\omega}}\right]$$

$$= \begin{cases} \overline{i\eta^3(\tau)} & \text{if} \quad w_2 = 1 \\ \frac{S}{\mathrm{Im}\tau\,\boldsymbol{\omega}}\overline{\vartheta_4(\tau)} & \text{if} \quad w_2 = 0 \end{cases} .$$

The u-plane integral

Theorem (Moore and Witten, 1997)

For $X = CP^2$, it follows

$$\mathbf{Z}_u(p,S) = \int_{H/\Gamma_0(4)}^{reg} \frac{d\tau \wedge d\bar{\tau}}{\mathrm{Im}\tau^{\frac{3}{2}-w_2}} \, e^{2\,p\,u+S^2\,\widehat{T}(u)} \frac{du}{d\tau} \frac{\Delta^{\frac{1}{8}}}{\omega^{\frac{3}{2}-w_2}} \begin{cases} \overline{\eta^3(\tau)} & \text{if} \quad w_2 = 1 \\ S\,\overline{\vartheta_4(\tau)} & \text{if} \quad w_2 = 0 \end{cases} .$$

with

$\vartheta_4(\tau) = \sum_{n=-\infty}^{\infty} (-1)^n \, q^{\frac{n^2}{2}}$, $\eta(\tau) = q^{\frac{1}{24}} \prod_{n\geq 1} (1-q^n)$.

Remarks about the u-plane integral

- integrand is modular invariant under $\Gamma_0(4)$;
- integrand has singularities at nodes at cusps $u = \pm 1, \infty$, regularization procedure must be applied, the cusps contributions are the only contributions to $\mathbf{Z}_u$:

  $$\mathbf{Z}_u(p,S) = \mathbf{Z}_u(p,S)\mid_{u=-1} + \mathbf{Z}_u(p,S)\mid_{u=1} + \mathbf{Z}_u(p,S)\mid_{u=\infty}$$

- Renormalization group action:

  $$x_\Lambda = \Lambda^2 x,\ y_\Lambda = \Lambda^3 y,\ g_{2,\Lambda} = \Lambda^4 g_2,\ g_{3,\Lambda} = \Lambda^6 g_3,$$

  $$u_\Lambda = \Lambda^2 u,\ \mathbf{a}_\Lambda = \Lambda\,\mathbf{a},$$

  $$T = \left(\frac{\partial u_\Lambda}{\partial \Lambda}\right)_{\substack{\Lambda = 1 \\ \mathbf{a}_\Lambda = \mathrm{const}}}$$

Results for $X = CP^2$

Proposition (M-Ono)

For $X = CP^2$ and $w_2 = 1$, it follows:

$$\mathbf{Z}_u(p,S)\,|_{u=\pm 1} = 0$$
$$\mathbf{Z}_u(p,S)\,|_{u=\infty} = \mathbf{Z}(p,S)$$

Proof

method: integration by parts using nonholomorphic modular form of weight $(\frac{1}{2}, 0)$ for $\Gamma_0(4)$:

- near cusp $u = \infty, \mathrm{Im}\tau \to \infty$ of type I_4^*, $\tau = x + iy$:

$$\mathbf{Z}_u \sim \int d\tau \wedge d\bar{\tau} \, \frac{\partial}{\partial \bar{\tau}} \Big(\dots \Big) = 2\, i \int_0^4 dx \, \Big(\dots \Big) \; ;$$

- integration involves $\eta(\tau)^3$ as divergence $\Rightarrow$ mock theta $Q(\tau)$;
- prove exponential convergence *after* $\int dx$, then compute

$$\mathbf{Z}_u = \lim_{y \to \infty} 2\, i \int_0^4 dx \, \Big(\dots \Big) = \sum_{m,n \geq 0} \mathbf{D}_{m,2n} \, \frac{p^m S^{2n}}{m!(2n)!}$$

- Gymnastics with heat operators and differential operators

Mock theta function

For $\eta^3(\tau)$ (modular form of weight $\frac{3}{2}$) and $q = e^{2\pi i\tau}$, $\tau = x + iy$, we look for solutions of

$$\frac{\partial}{\partial\bar{\tau}}\left[Q^+(\tau) + Q^-(\tau, y)\right] = \frac{1}{\sqrt{y}}\,\overline{\eta^3(\tau)}\,.$$

- $Q^+(\tau)$: mock modular form of weight $\frac{1}{2}$, holomorphic but not quite modular;
- $Q^-(\tau)$: correction term, non-holomorphic; has only neqative Fourier modes $\sim q^{-n}$; each Fourier coefficient has exponential in $y \to \infty$;
- $Q^+(\tau) + Q^-(\tau)$ is modular form for $\Gamma(2) \cap \Gamma_0(4)$;

Mock theta function

- $Q^+(\tau)=\frac{1}{q^{\frac{1}{8}}}\sum_{\alpha\geq 0}H_\alpha\, q^{\frac{\alpha}{2}}$;
- $Q^-(\tau)=\frac{1}{q^{\frac{1}{8}}}\sum_{\alpha\geq 0}H_{-\alpha}(y)\, q^{-\alpha}$ w/ $\lim_{y\to\infty}H_{-\alpha}(y)q^{-\alpha}=0$;
- u-plane integral:

$$\mathbf{Z}_u\sim\lim_{y\to\infty}\int_0^4 dx\,\left(\sum_{\beta\geq 0}C_\beta\, q^{\frac{\beta}{4}}\left[\,Q^+(\tau)+Q^-(\tau,y)\right]\right)$$

- since $\int_0^4 dx\, q^{\frac{\alpha}{4}}=4\,\delta_{\alpha,0}$ it follows:

$$\mathbf{Z}_u\sim\lim_{y\to\infty}\mathrm{Coeff}_{q^0}\left(\sum_{\beta\geq 0}C_\beta\, q^{\frac{\beta}{4}}\left[\,Q^+(\tau)+Q^-(\tau,y)\right]\right)$$

$$=4\,\mathrm{Coeff}_{q^0}\left(\sum_{\beta\geq 0}C_\beta\, q^{\frac{\beta}{4}}\,Q^+(\tau)\right)$$

Coefficients $D_{m,2n}$

- A gory and lengthy calculation gives:

$$\mathbf{D}_{m,2n} = \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{(-1)^{k+j+1}}{2^{n-2j-1}\, 3^{n-j}} \frac{(2n)!}{(n-k)!\, j!\, (k-j)!} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(j+\frac{1}{2}\right)}$$

$$\times \, \mathrm{Coef}_{q^0} \left[\frac{\vartheta_4^9(\tau) \, \left[\vartheta_2^4(\tau) + \vartheta_3^4(\tau)\right]^{m+n-k}}{\left[\vartheta_2(\tau)\, \vartheta_3(\tau)\right]^{2m+2n+3}} \, E_2^{k-j} \left(q \frac{d}{dq} \right)^j Q^+ \right] .$$

What does this have to do with $M(q)$?

- $Q^+(q) + Q^-(q)$ compatible with cusp width at singular points:

u	E_{sing}	Q^+
∞	I_4^*	$Q^+(\tau) = q^{-\frac{1}{8}}\left(1 + 28\, q^{\frac{1}{2}} + 39\, q + 196\, q^{\frac{3}{2}} + 161\, q^2 + \cdots\right)$
± 1	I_1	$Q^+(\tau_S) = \frac{1}{\sqrt{-i\,\tau_s}}\, Q^+\left(-\frac{1}{\tau_s}\right) = q^{-\frac{1}{8}}\left(\frac{5}{2} + \frac{111}{2}\, q + \frac{413}{2}\, q^2 + \cdots\right)$

- $Q^+(q^8)$ is given by Ramanujan's mock theta function $M(q)$:

$$4M(q^8) + \frac{28\eta(16\tau)^8}{\eta(8\tau)^7} + \frac{3\eta(8\tau)^5}{2\eta(16\tau)^4} + \frac{48\eta(32\tau)^8}{\eta(8\tau)^3\eta(16\tau)^4} - \frac{\eta(8\tau)^5}{2\eta(16\tau)^4}$$

The end game!

- We can now show that

$$Z(p,S) = Z_u(p,S)\,|_{u=\infty}\quad .$$

- For every m, n, just show that **zero is** the constant of

$$\sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^j\,\frac{(2n)!}{(n-k)!\,j!\,(k-j)!}\,\frac{\vartheta_4^8\,\left[\vartheta_2^4+\vartheta_3^4\right]^m}{\left[\vartheta_2\,\vartheta_3\right]^{2m+2n+3}}\;E_2^{k-j}$$
$$\times\Bigg[\frac{(-1)^{n+1}}{2^{k-3}\,3^k}\,\frac{(n-k)!}{(2n-2k)!}\,\left[\vartheta_2^4+\vartheta_3^4\right]^j\;F_{2(n-k)}$$
$$-\frac{(-1)^{k+1}}{2^{n-2j-1}\;3^{n-j}}\,\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(j+\frac{1}{2}\right)}\vartheta_4(\tau)\,\left[\vartheta_2^4(\tau)+\vartheta_3^4\right]^{n-k}\left(q\frac{d}{dq}\right)^j Q\Bigg]\;.$$

Freeman Dyson, 1987

"My dream is that I will live to see the day when our young physicists, struggling to bring the predictions of superstring theory into correspondence with the facts of nature, will be led to enlarge their analytic machinery to include mock theta-functions..."

$\Uparrow$

q-series