Dave Morrison:

Affine (resolved) conifold

(Nagao - Nakajima

cf Jafferis - Moore

Chuang - Jafferis (next week))

$xy - zt = 0$ "conifold" singularity

$\mathbb{C}^4 /\!/ u(1) = \mu^{-1}(a)/u(1)$

varying symplectic reduction

$a > 0$  $\mathbb{CP}^1$ Y^+

$a = 0$ conifold X

$a < 0$  $\mathbb{CP}^1$ Y^-

$\mathrm{vol}(\mathbb{CP}^1) \sim a$

$[\mathbb{CP}^1]_{a<0} = -[\mathbb{CP}^1]_{a>0}$

deformed conifold

$xy - zt = \varepsilon$

<u>coherent sheaves on Y^+</u>:

$\mathrm{Coh}(Y^+) \ni \mathcal{E}$

$\begin{cases} \mathrm{supp}\,\mathcal{E} \subseteq Y^+ \\ \text{wrap D-branes on } \mathrm{supp}\,\mathcal{E}, \\ \qquad \mathcal{E} = \text{gauge bdl.} \end{cases}$

<u>derived category</u>:

- boundary conditions in top. open string theory give $D^b(\mathrm{Coh}\, Y^+)$
- $D^b(\mathrm{Coh}\, Y^+) \cong D^b(\mathrm{Coh}\, Y^-)$

  $\cup\!\!|$ $\qquad\qquad\qquad \cup\!\!|$

  but $\mathrm{Coh}\, Y^+ \not\leftrightarrow \mathrm{Coh}\, Y^-$

(abelian category) $\rightsquigarrow$ derived category

Π-stability: in physical string theory can tell phy states from BPS mass formula

$D^b(\mathrm{Coh}\, Y^+) \supseteq$ (abelian cat of BPS states)

$$D^b(\mathrm{Coh}\, Y^+) \cong D^b(\mathrm{Coh}\, Y^-) \cong D^b(\text{mod-}\mathcal{A})$$ ③

"$a > 0$" "$a < 0$" modules of D-brane algebra, obtained from hag. description

"$a = 0$"

$\mathcal{A}$: quiver diagram

$$\begin{tikzcd} p_1 \arrow[r, bend right, "x_1"'] \arrow[r, bend right=60, "x_2"'] & p_2 \arrow[l, bend right, "y_2"'] \arrow[l, bend right=60, "y_1"'] \end{tikzcd}$$

$SU(n_1) \times SU(n_2)$

$$\mathbb{C}\langle x_1, x_2, y_1, y_2 \rangle \Big/ \left(\text{path relations, e.g. } x_1 x_2,\ y_1 y_2 + \frac{\partial W}{\partial \Phi_\alpha} \right)$$

$$W = (x_1 y_1 x_2 y_2 + \dots) - (x_1 y_2 x_2 y_1 + \dots)$$

$$\rightsquigarrow \quad y_1 x_2 y_2 = y_2 x_2 y_1$$

mod-$\mathcal{A}$ = category of reps of $\mathcal{A}$

$p_1 \rightsquigarrow V_1$

$p_2 \rightsquigarrow V_2$ data: $n_i = \dim V_i$

$a_i \in \mathrm{Hom}(V_1, V_2) \quad i = 1,2$

$b_j \in \mathrm{Hom}(V_2, V_1) \quad j = 1,2$

s.t. $b_1 a_2 b_2 = b_2 a_2 b_1, \ldots$

choices: $\{(V_1, V_2, a_1, a_2, b_1, b_2) \,\|\, \text{relat.}\}$

$\circlearrowleft\ \mathcal{U}(n_1) \times \mathcal{U}(n_2)$

$Gl(n_1, \mathbb{C}) \times Gl(n_2, \mathbb{C})$

<u>moduli spaces:</u>

$$\left. \underset{(b_1 a_2 b_2 = b_2 a_2 b_1, \ldots)}{\{(a_1, a_2, b_1, b_2)\}} \middle/ Gl(n_1, \mathbb{C}) \times Gl(n_2, \mathbb{C}) \right.$$

A. Kings: "moduli of representations of finite dimensional algebras" Quart. J. Math. Oxford 45 (1994), pp. 515–530

V = vector space

G = reductive group

$V/\!\!/G$ Defining using Geometric Invariants Theory (GIT) requires lifting G-action to a line bdl $\mathcal{L}$ on V

specify a character $\chi_\xi : G \to \mathbb{C}^*$
use χ for action on $\mathcal{L}$

$$\mathcal{L} = V \times \mathbb{C}$$

G on $\mathcal{L}$ $\quad g\ (v, z) \mapsto (g\cdot v, \chi(g)z)$

χ determined by $\Theta : K_0(\text{mod-}A) \to \mathbb{R}$
(additive fct)

$$[\xi] - [f]$$

for $0 \to \xi' \to \xi \to \xi'' \to 0$

$$\Longrightarrow [\xi] = [\xi'] + [\xi'']$$

$M \in Ob(\text{mod-}A)$ is called
<u>Θ-semistable</u> if
$\Theta(M) = 0$ and $\forall\, M' \subseteq M$ subobj.
$\Theta(M') \geq 0$

<u>Θ-stable</u>: if the only subobjects w/ $\Theta(M')=0$ are M and 0.

* <u>point of GIT</u>:

R = ring of G-inv fcts on $\mathcal{L}$ graded by fiber degree

$$\operatorname{Proj} R = \text{quotient of } V \text{ by } G\text{-action}$$

$$\begin{array}{l} \parallel \\ V^{ss} = \{(G,\chi)\text{-semistable pts}\} \Big/ (x \sim x' \text{ if closures of orbits intersect}) \\ \cup\!\!\mid \\ V^{s} = \{(G,\chi)\text{-stable pts}\} \Big/ (x \sim x \text{ if in same orbit}) \end{array}$$

(Θ determines a χ)

* <u>symplectic reduction</u>

$G \supseteq G_{\mathbb{R}}$

$G_{\mathbb{R}}$ cpct

$G_{\mathbb{R}} \otimes \mathbb{C} = G$

$$\mu : V \times \mathbb{C} \to \operatorname{Lie}(G_{\mathbb{R}})^*$$

$\chi : G \to \mathbb{C}^*$

$d\chi : \mathrm{Lie}(G) \to \mathrm{Lie}(\mathbb{C}^*)$

$d\chi|_{\mathrm{Lie}(G_{\mathbb{R}})} \ \mathrm{Lie}(G_{\mathbb{R}}) \to \mathrm{Lie}(u(1)) = i\,\mathbb{R}$

$\cancel{\mu((v,z))(d\chi) \in \mathbb{R}}$

$d\chi^* : \mathrm{Lie}(\mathbb{C}^*)^* \to \mathrm{Lie}(G)^*$

$d\chi^*|_{\mathrm{Lie}(G_{\mathbb{R}})} \ i\,\mathbb{R} \to \mathrm{Lie}(G_{\mathbb{R}})^*$

$d\chi = d\chi^*(1) \in \mathrm{Lie}(G_{\mathbb{R}})^*$

<u>Thm</u>: $\mu^{-1}(d\chi)$ meets each G orbit which is closed in V^{ss} in a unique $G_{\mathbb{R}}$-orbit and no other G-orbit.

Hence $\mu^{-1}(d\chi)/G_{\mathbb{R}} = \mathrm{Proj}\, R$

* <u>Nagao, Nakajima</u>:

Moduli of pairs

(V, v) w/ $v \in V_0$

data $V_0, V_1, \ldots$

(V, v) is ζ-(semi)stable iff

1) $0 \neq V' \subseteq V \Rightarrow \zeta_0 \dim V'_0 + \zeta_1 \dim V'_1 \underset{(\leqq)}{<} 0$

and 2) $V' \subseteq V$ s.t. $v \in V'_0$

$\Rightarrow \zeta_0 \operatorname{codim} V'_0 + \zeta_1 \operatorname{codim} V'_1 \underset{(\geqq)}{>} 0$

moduli spaces:

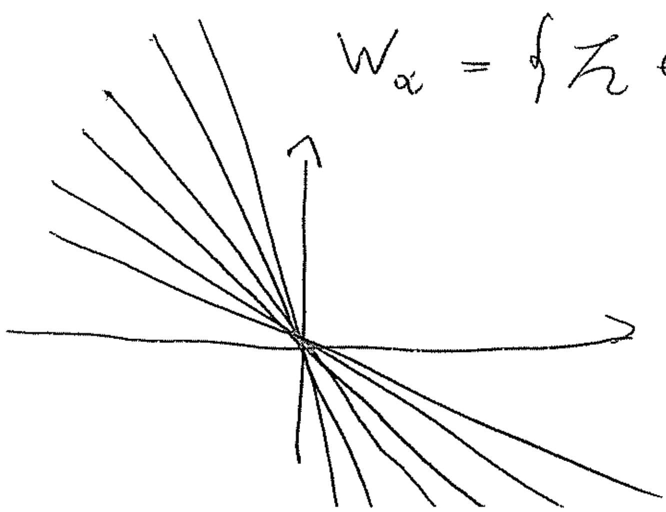
$\mathfrak{M}_\zeta(n_1, n_2)$

each of them has virtual fund. class

$$Z_\zeta(q_1, q_2) = \sum_{(n_1, n_2)} \left(\int_{[\mathfrak{M}_\zeta(n_1, n_2)]^{vir}} 1 \right) q_1^{n_1} q_2^{n_2}$$

$\mathbb{R}^2 \setminus \bigcup_\alpha W_\alpha$ $\qquad \alpha = (1,1)$, $\underset{m \geq 0}{(m, m+1)}$, $\underset{m > 0}{(m, m-1)}$

$W_\alpha = \{\zeta \in \mathbb{R}^2 \mid \zeta \cdot \alpha = 0\}$



$m_{\xi}(n_1, n_2)$ are const for

$\xi \in$ connected comp. of

$$\mathbb{R}^2 \setminus \bigcup_{\alpha} W_{\alpha}$$

But they jump crossing a wall.