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Langlands Program and Mirror Symmetry, II

Number Theory — Curves over finite fields — Riemann surfaces — Physics

$\underbrace{\text{Number Theory} \quad \text{Curves over finite fields}}_{\text{classical Langlands program}}$

$\underbrace{\text{Riemann surfaces} \quad \text{Physics}}_{\text{S-duality and mirror symmetry}}$

$\underbrace{\text{Curves over finite fields} \quad \text{Riemann surfaces}}_{\text{geometric Langlands program}}$

(since 80s: Deligne, Drinfeld, Beilinson, Laumon, ...)

X = compact Riemann surface (smooth projective algebraic curve / $\mathbb{C}$)

G = complex reductive Lie group (e.g., $GL_n(\mathbb{C})$)

${}^LG$ = Langlands dual group (e.g.: n even; $G = SL_n/\mathbb{Z}_2$, ${}^LG = SL_n/\mathbb{Z}_{n/2}$)

G. L. Correspondence (unramified)

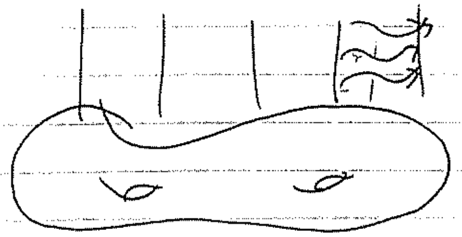
$$\{\pi_1(X) \longrightarrow {}^LG\} = \{(E, \nabla),\ E = \text{principal holomorphic (algebraic) } {}^LG\text{-bundle on } X,\ \nabla = \text{holomorphic (algebraic) connection on } E\}$$

$\uparrow$ geometric analogue of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$

$$\nabla = \nabla_{\frac{\partial}{\partial z}} \quad (1,0)$$

holo. structure determines $\nabla_{\frac{\partial}{\partial \bar{z}}}$ $(0,1)$

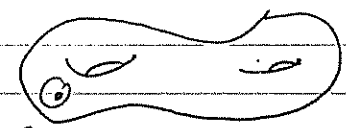
$\Rightarrow$ flat connection



LHS: $\{(E,\nabla), \ldots\}$ $\longleftrightarrow$ RHS: $\left\{ \begin{array}{c} \text{Hecke eigensheaves} \\ \text{on } \underline{Bun_G} \end{array} \right\}$

moduli space of G-bundles on X
(algebraic stack)

$G(\mathbb{A})$



F = field of meromorphic functions on X

$$F \hookrightarrow F_x = \mathbb{C}((t_x)) \supseteq \mathcal{O}_x = \mathbb{C}[[t_x]]$$

$$\mathbb{A} = \prod{}' F_x$$

$$\mathcal{O} = \prod \mathcal{O}_x$$

$$G(F) \hookrightarrow G(\mathbb{A})$$

$$G(F)\backslash G(\mathbb{A}) \circlearrowleft G(\mathbb{A}) \text{ acts}$$

$X/\mathbb{F}_q$: $\quad \underset{\circlearrowleft\, G(\mathbb{A})}{Fun\left(G(F)\backslash G(\mathbb{A})\right)} = \underset{\text{discrete spectrum}}{\bigoplus \pi} \oplus \underset{\text{cts spectrum}}{\int \pi}$

for now, only look at unramified reps: $v \in \pi$, $G(\mathcal{O})\cdot v = v$.

Then $v \rightsquigarrow$ right $G(\mathcal{O})$ invariant function on $G(F)\backslash G(\mathbb{A})$

$= $ function on $G(F)\backslash G(\mathbb{A})/G(\mathcal{O})$ <u>Hecke eigenfunction</u>

<u>A Weil</u> $G(F)\backslash G(\mathbb{A})/G(\mathcal{O})$ = set of equivalence classes of G-bundles on X.

$\rightsquigarrow \mathrm{Bun}_G$

Automorphic function $\xrightarrow[\text{geometric story}]{\text{in the}}$ a $\mathcal{D}$-module on Bun_G (system of linear PDE's)

(Hecke eigensheaves)

On the category of $\mathcal{D}$-modules on Bun_G we have natural functors:

Hecke functors $H_{V,x}$ $\quad V \in \mathrm{Rep}\,{}^LG$, $x \in X$

$\mathcal{F}$ = Hecke eigensheaf

$$H_{V,x}(\mathcal{F}) = \underline{V} \otimes \mathcal{F}$$

$$\mathcal{E} = (E, \nabla) \longmapsto \mathcal{F}_{\mathcal{E}}$$

$\mathcal{Y}({}^LG, X)$, moduli space of flat ${}^LG$-bundles on X

$$\mathrm{Coh}\left(\mathcal{Y}({}^L G, X)\right) \overset{\text{for derived categories}}{\simeq} \mathcal{D}\text{-mod}(\mathrm{Bun}_G)$$

category of coherent sheaves ($= \mathcal{O}$-modules) on $\mathcal{Y}({}^L G, X)$

category of $\mathcal{D}$-modules on Bun_G.

$W_{V,x}$ Wilson operators

$H_{V,x}$ Hecke operators

Ψ

$\mathcal{E} \longmapsto \mathcal{O}_{\mathcal{E}}$ skyscraper sheaf $\longleftrightarrow \mathcal{F}_{\mathcal{E}}$

obviously eigensheaves $\longrightarrow$ therefore, these should also be

For $G = \mathbb{C}^*$, this is a theorem. (this is true!)

Kapustin-Witten: SYM theory on $X \times \Sigma$ gauge group G

another Riemann surface

Limit when X becomes small

Get 2-dim'l sigma model on Σ

$$\Sigma \to \mathcal{M}_H(G)$$

I

$$\mathcal{M}_H(G) = \{ (E, \phi) : \begin{array}{l} E = G\text{-bundle} \\ \phi = \text{Higgs field} : \phi \in \Gamma(X, \mathfrak{g}_E \otimes K_x) \end{array} \}$$

$\mathfrak{g}_E = E \times_G \mathfrak{g}$

K_x: canonical line bundle

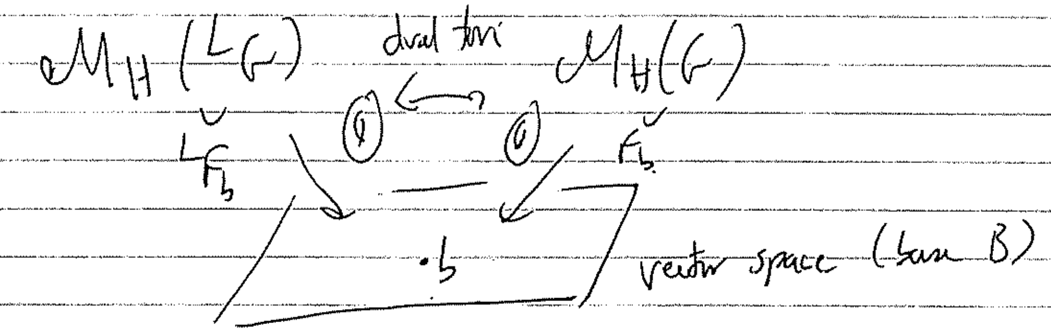
hyperKähler manifolds I, J, K

$$\overset{J}{\parallel}$$

$$\mathcal{Y}({}^L G, X) = \{ (E, \nabla) \}$$

Theory for $G \longrightarrow \mathcal{M}_H(G)$

$ {}^LG \longrightarrow \mathcal{M}_H({}^LG)$



${}^LF_b$ = moduli space of line bundles on F_b with a flat unitary connection

B-model, cx str J:

$\mathcal{M}_H({}^LG) = \mathcal{Y}({}^LG, X)$.

category of B-branes is just $\mathrm{Coh}(\mathcal{Y}({}^LG, X))$

$\xleftrightarrow{\text{equivalence}}$ mirror symmetry

A-model, cx str I:

$\mathcal{M}_H(G) = T^* \mathrm{Bun}_G$, has a symplectic form ω.

Cat. of ~~A~~ A-branes = Fukaya category

e.g. $(\mathcal{L}, \nabla)$, $\mathcal{L}$ = Lagrangian subvariety

∇ = flat connection

$$\boxed{\mathrm{Coh}\left(\mathcal{Y}({}^L G, X)\right)} \xleftrightarrow{\ GLC\ } \boxed{\mathcal{D}\text{-mod}(\mathrm{Bun}_G)}$$

$W_{V,x}$ $\circlearrowright$ $\quad \ni \mathcal{O}_\mathcal{E} \qquad\qquad \mathcal{F}_\mathcal{E} \in \quad \circlearrowleft H_{V,x}$

$$\| \qquad\qquad\qquad\qquad \wr\|$$

$$\boxed{\begin{array}{c}\text{B-brane on } \mathcal{M}_H({}^L G) \\ (\text{in Cx. str. } J)\end{array}} \xleftrightarrow[\text{symmetry}]{\text{mirror}} \boxed{\text{A-brane on } \mathcal{M}_H(G) = T^*\mathrm{Bun}_G}$$

$W_{V,x}$ $\circlearrowright$ $\quad \ni \mathcal{O}_\mathcal{E} \qquad\qquad \ni A_\mathcal{E} \qquad\qquad \circlearrowleft T_{V,x}$

('t Hooft operators)

$$\mathcal{D}\text{-mod}(Z)$$

$$\wr\|$$

$$\text{A-brane on } T^*Z$$

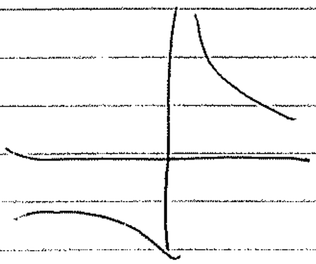
${}^L\mathcal{F}_b$ $\quad \mathcal{E}$ $\qquad F_b$

$\mathcal{E} \in {}^L F_b \longmapsto (F_b, \nabla_\mathcal{E}$ = flat unitary line bundle on F_b)

$$\| \\ A_\mathcal{E}$$

$\cdot\, b \in \mathrm{Bun}_G$

char. var.



$(Z\partial_z - 1)$ $\quad 2p' = 0$

B_{cc}

$\circlearrowright$

$B \Rightarrow \mathrm{Hom}(B_{cc}, B)$