Kähler - Ricci flow on Hirzebruch surfaces

(jt w/ Jian Song)

Outline:

- Background
- Hirzebruch surfaces
- Higher dim. analog
- Calabi ansatz
- Estimates

Background

(M, ω_0) cpct Kähler

$\dim_{\mathbb{C}} M = n$

$g^0_{i\bar{j}} > 0$ Hermitian

$$\omega_0 = \sqrt{-1}\, g^0_{i\bar{j}}\, dz^i \wedge dz^{\bar{j}}$$

Kähler: $d\omega_0 = 0$

Q1: Does $\exists$ Kähler $\omega \in [\omega_0] \in H^{1,1}(M;\mathbb{R})$ w/ $Ric(\omega) = \lambda\,\omega$, $\lambda = -1, 0, 1$?

Q2: Does the Kähler - Ricci flow (normalized)

$$\begin{cases} \frac{\partial}{\partial t} \omega = -\mathrm{Ric}(\omega) + \lambda \omega \\ \omega(0) = \omega_0 \end{cases}$$

converge to a Kähler - Einstein metric ?

since $c_1(M) \overset{\text{defn}}{=} [\mathrm{Ric}(\omega_0)] \in H^{1,1}(M;\mathbb{R})$

$\mathrm{Ric}(\omega_0) = \sqrt{-1}\,\partial\bar{\partial} \log \det g^0$

& $\exists$ KE $\Rightarrow$ $c_1(M) < 0$, $c_1(M) = 0$, $c_1(M) > 0$

- $c_1(M) < 0$, $\lambda = -1$ (Yau, Aubin)
  (1) $\exists!$ KE metric $\omega \in -c_1(M)$
  (2) yes, flow converges (Cao)
  & $\omega_0 \in -c_1(M)$
- $c_1(M) = 0$, $\lambda = 0$ (Yau)
  (1) (Yau) $\exists!$ ω in every Kähler class
  (2) (Cao) yes, flow converges

- $c_1(M) > 0$

(1) in general no

(2) (Perelman, Tian-Zhu):

If $\exists$ KE metric then KRflow converges to a KE

Q: · If $\nexists$ KE metric, what happens to KR flow?

- Does it find the best metric?
- Does the flow "simplify" the mfld?

$c_1(M) \leq 0$: · K_M big and nef

$$c_1(M) = -c_1(K_M)$$

- big: $\int_M (-\operatorname{Ric}(\omega_0))^n > 0$

- nef: $\forall C$ curves $\int_C (-\operatorname{Ric}\,\omega_0) \geq 0$

$\Rightarrow$ KRF exists $\forall t$ and converges to a sing KE metric whic is smooth outside a divisor (Tsuji, Tian-Zhang)

- K_M nef, $n=2$, $\mathrm{Kod}(M)=1$
  $\dim H^0(M, K^N) \sim N$ as $N \to \infty$

KRF converges weakly to a "generalized KE metric" on Σ (Song–Tian)

Hirzebruch surfaces

$k = 0, 1, 2, \ldots$

$$M_k = \mathbb{P}(\underbrace{H^k \oplus \mathbb{C}_{\mathbb{P}^1}})$$

- rank 2 vct-bdl over $\mathbb{P}^1$
- H = hyperplane bdl.
  $\mathbb{C}_{\mathbb{P}^1}$ = trivial bundle

$\mathbb{P}^1$ – bdl over $\mathbb{P}^1$

- all rational surfaces are blow-ups of $\mathbb{P}^2$ or M_k

- $k=0$: $\mathbb{P}^1 \times \mathbb{P}^1$ ← ignore
- $k=1$: $\mathbb{P}^2$ blow up at 1-pt

$\Big\}$ $c_1(M) > 0$

- $k=2$: $c_1(M) \geq 0$
- $k \geq 3$: $c_1(M)$ indefinite

Kähler cone:

Every α Kähler class can be written

$$\alpha = \frac{b}{k}[D_\infty] - \frac{a}{k}[D_0]$$

$$0 < a < b$$

hol. curves

D_∞ = image of section $(0,1)$ in M_k

section of $\mathbb{C}_{\mathbb{P}^1}$

section in H^k

D_0 = $\overline{\text{image of } (\sigma, 0)}$

σ any hol section of H^k

Consider Kähler - Ricci flow (unnormal.)

$$\begin{cases} \frac{\partial}{\partial t}\omega = -\operatorname{Ric}(\omega) \\ \omega(0) = \omega_0 \in \alpha_0 \end{cases}$$

$$= \frac{b_0}{k}[D_\infty] - \frac{a_0}{k}[D_0]$$

Assumption: ω_0 is invariant under $G_K \cong U(2)/\mathbb{Z}_k$ maximal cpt subgp of $\operatorname{Aut}(M)$

What happens to

$$\alpha_t = \frac{b_t}{k}[D_\infty] - \frac{a_t}{k}[D_0]$$

$$-[\mathrm{Ric}(\omega)] = c_1(M_k) = \frac{k+2}{k}[D_\infty] + \frac{k-2}{k}[D_0]$$

$$\Rightarrow \quad \partial_t \alpha_t = -\frac{k+2}{k}[D_\infty] - \frac{k-2}{k}[D_0]$$

$$b_t = b_0 - (k+2)t$$
$$a_t = a_0 + (k-2)t$$

<u>Thm</u> (Song, W) $\omega(t)$ solve KRF on M_k

a) $k \geq 2$ (KRF) exists on $[0,T)$ where $T = \dfrac{b_0 - a_0}{2k}$ and

as $t \to T$: $(M_k, g_k(t)) \xrightarrow{GH} (\mathbb{P}^1, a_T\, g_{FS})$

Fubini-Study metric on $\mathbb{P}^1$

(b) $K = 1$ Three subcases

i) $3a_0 = b_0$ (KRF) exists until $T = a_0$ and

$$t \to T$$

$$(M_1, g(t)) \xrightarrow{GH} \{pt\}$$

ii) $3a_0 > b_0$. The same as (a) above.

iii) $3a_0 < b_0$: The flow exists on $[0 \ \ T)$ $T = a_0$

and as $t \to T$, $g(t) \to g_T$ smoothly on cpct subsets of $M - D_0$.

contracting of D_0

If $(\overline{M}, d_T)$ is metric completion of $(M - D_0, g_T)$ then

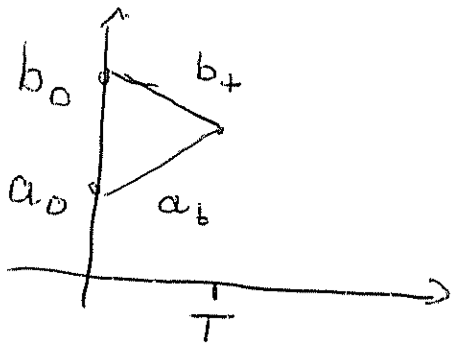
$$(M_1, g(t)) \xrightarrow{GH} (\overline{M}, d_T)$$

$\cong$ $\mathbb{P}^2$ homeo

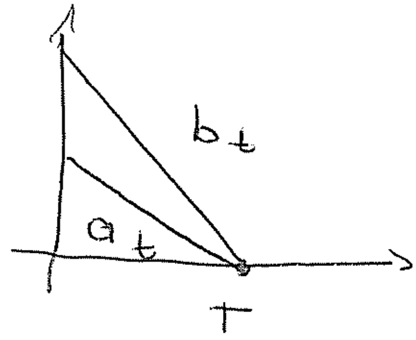
+ has finite diameter

(a)

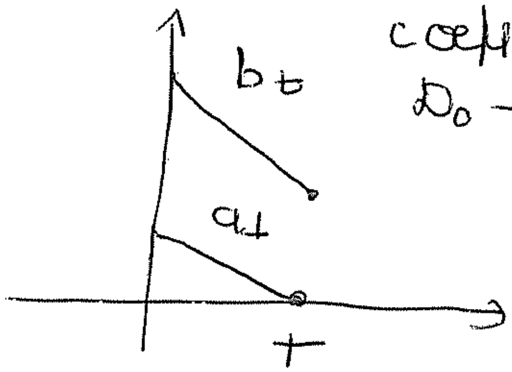
(b) ii)



(b) i)



(b) (iii)



coeff of $D_0 \to 0$

<u>Rk</u>: Conjectured by Feldman-Ilmanen-Knopf

Higher dims

$n \geq 2$

$$M_{n,k} = \mathbb{P}(\underbrace{H^k \oplus \mathbb{C}_{\mathbb{P}^1}}_{\text{rank 2 bdl over } \mathbb{P}^{n-1}})$$

$\uparrow$

$\mathbb{P}^1$ - bdl over $\mathbb{P}^{n-1}$

<u>Thm</u>: (Song, W)

(a) $k \geq n$ $\quad (M_k, g(t)) \xrightarrow{GH} (\mathbb{P}^{n-1}, a_T g_{FS})$

(b) $1 \leq k \leq n-1$

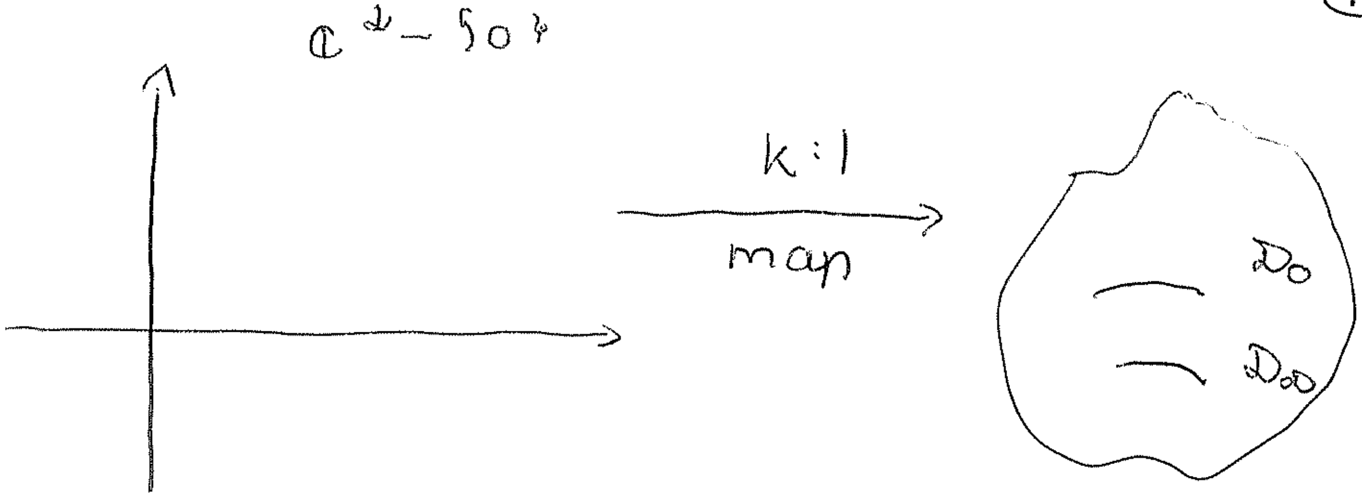
i) $a_0(n+k) = b_0(n-k)$ $\quad (M, g(t)) \overset{GH}{\rightsquigarrow} pt$

ii) $a_0(n+k) > b_0(n-k)$ $\quad$ Same as a)

iii) $a_0(n+k) < b_0(n-k)$

$$(M_k, g(t)) \longrightarrow (\overline{M}, d_T)$$

$\mathbb{P}^n / \mathbb{Z}_k$ orbifold



$$u(z) \text{ on } \mathbb{C}^2 \quad \longleftrightarrow \quad G_k = u(z) / \mathbb{Z}_k$$

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} u \qquad u = u(|x_1|^2 + |y_1|^2)$$

$$\rho = \log(|x_1|^2 + |x_2|^2)$$

$$\rightsquigarrow \quad \frac{\partial u}{\partial t} = \log u'' + \log u' - 2\rho$$