Plan:

1) curves of genus 1, 2
2) elliptic surfaces over $\mathbb{CP}^1$
3) Kummer surfaces from SW curves

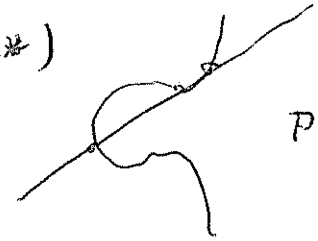
1) $\underline{g=1}$ E: elliptic curve

analy $E \cong \mathbb{C}/\Lambda$ $\Lambda = \mathbb{Z} + \tau\,\mathbb{Z}$ $\operatorname{im}(\tau) > 0$

$E' \cong E \iff \Lambda = \alpha(\Lambda')$ for some $\alpha \in \Lambda'$

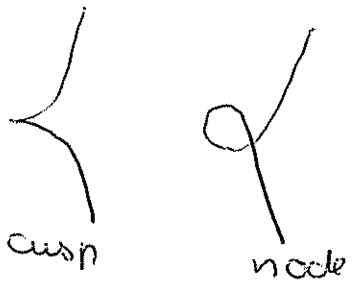
alg:

$E: \quad y^2 = x^3 + Ax + B \quad (*)$



$P + Q + R = O$

- $\Delta = 4A^3 + 27B^2 \neq 0$



$2P + O = O$

- $j = \dfrac{g_2^3}{\Delta}$
- only freedom in $(*)$

$x \mapsto x u^2$
$y \mapsto y u^3$
$g_2 \mapsto g_2/u^4$
$g_3 \mapsto g_3/u^6$
$j \mapsto j$

Thm. $E \cong E' \Rightarrow j = j'$

$\Leftarrow$ ~~$\left(\frac{g_2}{g_2'}\right) = k^4$~~ for arbitr field K

~~$\left(\frac{g_3}{g_3'}\right) = k^6$~~

$k \in K$

E

quadratic twist

$E^{(d)}$

$\frac{E}{\tau}$: $y^2 = x^3 + p x^2 + q x$

$P : (y, x) = (0, 0)$: 2-torsion pt

$E' = E/\{O; P\}$ $\quad y^2 = x^3 - 2p x^2 + (p^2 - 4q) x$

$\tau' = \tau/2$

over $\mathbb{C}$ rescale $q = 1$

$\rightsquigarrow$ $N_f = 0$ SW curve $\quad y^2 = x^3 + u x^2 + x$

$N_f = 2$ SW-cu $\quad y^3 = x^3 - 2u x^2 + (u^2 - 4) x$

<u>$g = 2$</u>

hyperelliptic

$y^2 = f(x)$ $\quad f$ sextic

Clebsch def igusa-clebsch inv $\quad [I_2; I_4; I_6; I_{10}] \in \mathbb{WP}^3_{2;4;6;10}$

$\cap$

$\mathbb{Z}[a_0, \dots, a_6]$

$C \cong C' \iff \exists\, r \in \mathbb{C}^*: I_d(f') = r^d I_d(f)$

$\rightsquigarrow$ 3 absolute inv $\quad i_1 = \frac{I_4}{I_2^2}, \; i_2 = \frac{I_6}{I_2^3}, \; i_3 = \frac{I_{10}}{I_2^5}$

2) $S \xrightarrow{\pi} C$ (for us $\mathbb{CP}^1$) elliptic fibration

alg surface S; section σ : $C \to S$, $\pi \circ \sigma = id$; $\pi^{-1}(pt) \cong E$

at least one section ⇝ Jacobian ellipt fibrat

~~described by~~ $y^2 = x^3 + Ax + B \; (*)$

$A \in k(C)$

$B \in k(C)$

group sections (w/ additn) $MW(\pi)$

finitely generated $MW(\pi) = MW(\pi)_{tor} \oplus \mathbb{Z}^r$

Q : how does (*) define S

- $C^0 = C - \{ pts \text{ whr } \Delta = 0 \}$

fill in suitable sing fibers at pts omitted

all sing fibrs <u>were</u> classified by Kodaira (determ by monodromy around sing)

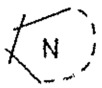
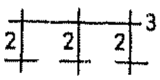
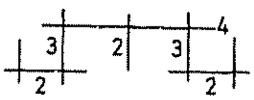
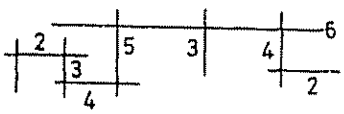
- surface smooth everywh
⇝ all fibers are <u>irreducible</u>
sing fibr are eithr nodes or cusps

 I_1  II

- surface not smooth, we have to take min resolutn
sing fibre is <u>reducible</u>
all irred comp

reducible fibr ⇘ I_n $(n \geq 2)$ I_n^* $(n \geq 0)$
III, IV

Table (6.1). Fibres of rational elliptic surfaces with section $y^2z=x^3+Axz^2+Bz^3$; $D=4A^3+27B^2$; $J=4A^3/D$

Name	Graph	$v_q(A)$	$v_q(B)$	$v_q(D)$	J
I_0		0	0	0	$\neq 0, 1, \infty$
I_0		0	K	0	1
I_0		L	0	0	0
I_1		0	0	1	∞
I_N		0	0	N	∞
I_0^*		2 $L\geqq 3$ 2	3 3 $K\geqq 4$	6 6 6	$\neq 0, 1, \infty$ 0 1
I_N^*		2	3	$N+6$	∞
II		$L\geqq 1$	1	2	0
III		1	$K\geqq 2$	3	1
IV		$L\geqq 2$	2	4	0
IV^*		$L\geqq 3$	4	8	0
III^*		3	$K\geqq 5$	9	1
II^*		4	5	10	0

j fct det JES up to quadratic twist

$g_2 \leftrightarrow g_2 t^2$
$g_3 \leftrightarrow g_3 t^3$
$I_n \leftrightarrow I_n^*$
$II \leftrightarrow IV^*$
$III \leftrightarrow III^*$
$IV \leftrightarrow II^*$

{ table terminates for $\nu(A) \geq 4$ and $\nu(B) \geq 6$ by minimalizing WEqn.

$g_2 \to \frac{g_2}{t^4}$ $g_3 \to \frac{g_3}{t^6}$

$C = \mathbb{C}P^1$ $[U\ V] \in \mathbb{C}P^1$ $u = \frac{U}{V}$

globally minimalized WEqn

$\deg A = 4n$
$\deg B = 6n$
$\deg \Delta = 12n$

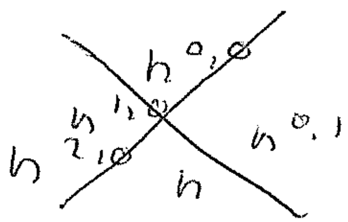
($n = 0$ $E \times \mathbb{C}P^1$)

$n = 1$ rational elliptic surface
(Thm: $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P^1} = \frac{1}{2} K3$

$n = 2$ elliptic $K3$ Δn

$n > 2$ honestly elliptic n

$n \geq 1$



$u = -1, 1\ \infty$

$N_f = 0$ sing fibers $2I_1 + I_4^*$ $\overset{2:1}{- - - \to}$ $N_f = 2$ $2I_2 + I_2^*$
$MW(\pi) = \mathbb{Z}_2$ translation by 2-torsion sech $MW(\pi) = \{\mathbb{Z}_2^2\}$

$g_2 (u-1)^2$
$g_3 (u-1)^3$

$N_f = 3$ $\underset{MW(\pi)}{I_1} + \underset{= \mathbb{Z}_4}{I_4} + I_1^*$

b) $n = 2$ elliptic K3 - surface

special: Kummer surfaces.

A: abelian surface

$-I$: 16 points

$Y = \widehat{A/\{I,-I\}}$: K3 surface

a) $A = E_\tau \times E_\sigma$ $y_i^2 =$

$\rightsquigarrow$ describes Y: Y: (more) quartic in $\mathbb{P}3$

$\mathbb{C}(x_1, y_1, x_2, y_2)$ inv under γ: $y_i \to y_i$, $x_i \to x_i$

var $t = y_1 y_2$ x_1 x_2

on Y you have elliptic fibrat., e.g. use $x_1 = u$ as n $4I_0^*$

$t^2 = f_1(x_1) f(x_2)$ MW (Γ) $(\mathbb{Z}_2)^2$

11 types of elliptic fibratio (classifie by Oguiso.

g_2, g_3 coeff are fcts of $j(\tau)$ $j(\sigma)$

b) $A = \mathrm{Jac}(C)$

quartic in $\mathbb{P}3$

16 nodes each node $6^{?}$, tog contains in

16 tropes each trope contain 6 nods

elliptic $\rightsquigarrow$ fibratia g_2, g_3 $i_1(f), i_2(f)$ $i_3(f)$

c) X: K3 surface

ι Nikulin inval: $\iota^* \omega = 0$

$\widehat{X/\{id, \iota\}}$: K3 surface Y

i Shioda Inose if Y is Kummer

$$X \overset{2:1}{- - \to} Y$$

$$\pi_* \quad T_X(2) \simeq T_Y$$

3) Kummer surfaces for SW curves

cases

a) $N_f = 2$ SW : X : sing fib $2I_2 + I_2^*$

$MW = (\mathbb{Z}_2)^2$

$h = (u-a)(u-b)(u-c)$ $\deg g_A \sim 2$

$\deg g_B = 3$

$\deg \Delta = 4$

$3I_2 + 3I_0^*$

$MW = (\mathbb{Z}_2)^2$ $\longrightarrow \deg h^2 g_2 \simeq 8$

$\deg h^3 g_3 = 12$

$a \neq \{\pm 1, \infty\}$

Q : Shioda-Inose str ?

A : yes but not given by translat of order two

$Y = X/\{id, i\}$ $\leadsto$ $\left.\begin{matrix} a \\ b \\ c \end{matrix}\right\}$ given cross two at i_1, i_2, i_3

$= Kum(Jac(C))$

base change $\quad s = \sqrt{\frac{t-b}{t-c}}$

$$\begin{array}{ccc} X \times_C B & \xrightarrow{\pi_Y} & B \\ \downarrow & & \downarrow \\ X & \xrightarrow{\pi_X} & C \end{array} \qquad t = \frac{s^2 c - b}{s^2 - 1}$$

effects depend on local ramification rel

- $t = 0, 1, \infty$ unramified pts

  $I_2 \longleftrightarrow 2 I_2$ in S

- $t = b \longleftrightarrow s = 0 \qquad (t-b) \sim s^2 + \dots$

  $c \longleftrightarrow s = \infty$

- minimalize W Eq.

$$g_2 \longrightarrow \frac{1}{\lambda^4} g_2$$

$$g_3 \longrightarrow \frac{1}{\lambda^6} g_3$$

(λ: polyn in s but not in t)

$X: \; 3I_2 + 3I_0^* \xrightarrow{\text{base chg}} Y: \; 6I_2 + 2I_0^*$

$MW(\pi) = \mathbb{Z} \qquad MW(\pi) = (\mathbb{Z}_2)^2 \oplus \mathbb{Z}$

b) do change base first $u = \dfrac{s^3 + a^2 s^2 + bs + c}{s^2}$

$u = \pm 1$ unram pts
$u = \infty$

$N_f = 0$: X : $2I_1 + I_4^*$ base chang Y : $6I_1 + I_4^* + I_8$
$MW(\pi) = \mathbb{Z}_2$ $MW = \mathbb{Z}_2$

transl by 2-torsion transl by 2-tors

$N_f = 2$: $2I_2 + I_2^*$ base chg $\longrightarrow$ Y : $6I_2 + I_2^* + I_4$
$MW(\pi) = \mathbb{Z}_2$ $MW = (\mathbb{Z}_2)^2$

$= \text{Ker}(\text{Jac}(C))$

a, b, c fcts of u_1 u_2 u_3

$a = 0$

X' : $2II^* + 8I_1$ $\longleftrightarrow$ ~~X : $6I_1 + I_{12}^*$~~
$MW = id$ $MW = \mathbb{Z}_2$
$NS = H + E_8^2$ $NS = H + D_{16}^+$

~~$II^* + 2I_1 + 2I_0^*$~~ Y .
~~$MW = id$~~
~~NS~~ . . .