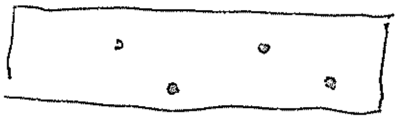
Morrison: Introduction to F-theory (1)
(4/30/10)

F-theory is a construction in string theory which uses a multiply defined function w/ $SL(2;\mathbb{Z})$ action.



$B \supset \Delta$

$b \in B \setminus \Delta$, $\tau(b)$ multiple-valued

$\pi_1(B \setminus \Delta) \to SL(2;\mathbb{Z})$

simple valued fn on $\widetilde{B \setminus \Delta}$

Alg geometry

E: elliptic curve, genus 1

$P \in E / \mathbb{C} \implies E \cong \mathbb{C} / \mathbb{Z} \oplus \tau \mathbb{Z}$

$P \mapsto 0$

$\tau \in \mathbb{C} \setminus \mathbb{R}$ (usually $\tau \in \mathfrak{h}$)

$\wp(z)$ Weierstraß $\wp$-fct

$z \in \mathbb{C}$, double periodic, meromorphic

$\underset{x}{\underset{\|}{\wp(z)}}, \underset{2y}{\underset{\|}{\wp'(z)}} \rightsquigarrow y^2 = x^3 + fx + g$

f, g functions of z

$Sl(2;\mathbb{Z})$ invariant

$\left\{ \begin{array}{l} Gl(2,\mathbb{Z}), z \in \mathbb{C} \setminus \mathbb{R} \\ Sl(2,\mathbb{Z}), z \in \mathfrak{h} \end{array} \right\}$

isomorphism class of curve: $j = \dfrac{1728 \cdot 4 f^3}{4f^3 + 27 g^2}$

$$\begin{array}{c} \mathcal{E} \\ \downarrow \\ B \end{array}$$ family of elliptic curves

j is a well-defined fct on B, **but** knowledge of j does not determine functions

Note: theory makes sense over arbitrary bases even Spec $\mathbb{Z}$, i.e. $y^2 = x^3 + fx + g$, $f, g \in \mathbb{Q}$

Families of elliptic curves

Kodaira early 1960's
(Néron over Spec $\mathbb{Z}$)
(assume char $k \neq 2, 3$)

Step 1:

$$\begin{array}{ccc} \mathcal{E} & \subseteq & \overline{\mathcal{E}} \\ \downarrow & & \\ B \setminus \Delta & \subseteq & B \end{array}$$

non-sing only

assume $\begin{array}{c} \mathcal{E} \\ \downarrow \\ B \setminus \Delta \end{array}$ has section

dim $B = 1$

a) $\overline{\mathcal{E}}$ birationally modify such that

$$\begin{array}{c} \hat{\mathcal{E}} \\ \downarrow \\ B \end{array}$$ is non-singular total space (assume minimal surface)

b) $\hat{\mathcal{E}}$ to W: contracting all curves of sing fibres not meeting section to points (singularity)

③

$$W \cong \{ y^2 z = x^3 + f x z^2 + g z^3 \} \subseteq \mathbb{P}^2_B$$

$\mathcal{L}$ on B, $f \in H^0(\mathcal{L}^{\otimes 4})$, $g \in H^0(\mathcal{L}^{\otimes 6})$

$$\mathbb{P}_B^2 = \mathbb{P}(\underset{z}{\mathcal{O}} \oplus \underset{x}{\mathcal{L}^{\otimes 2}} \oplus \underset{y}{\mathcal{L}^{\otimes 3}})$$

<u>dim $B = 1$</u>

Use structure of singularities of W to classify singular fibres.

(Kodaira did not assume $\exists$ a section, got a few "extra" items in classification) (e.g. multiple fibres

$$y^2 = x^3 + f x + g$$

local form of f, g, $\Delta = 4f^3 + 27g^2$

$\to B$, local coordinate s

- if surface is singular then fiber is singular, translate x, y to put sing at $(0,0)$

$$y^2 = x^3 + a_2 x^2 + a_4 x + a_6$$

singular $\rightsquigarrow$ $s | a_4$ and $s | a_6$

nonsingular $\iff$ $\Delta \neq 0$

W singular at $(0,0)$ $\iff$ $s^2 \mid a_6$

if $s^2 \mid a_6$: leading term

$$y^2 = a_2 x^2 + s\, a_{4,1} x + a_{6,2} s^2$$
$$+ \text{ higher order terms}$$

<u>blow-up</u>: quadratic terms ~~rank ≥ 2~~ unless $s \mid a_2$

⋮

$$y^2 = a_2 x^2 + s^k a_{4,k} x + a_{6,l} s^l$$
$$k \geq 2,\ l \geq 3$$

$s \mid a_2 \iff s \mid f$ and $s \mid g$

if $y^2 = a_2 x^2 \rightsquigarrow$ tangent directions $y = \pm \sqrt{a_2}\, x$

<u>Kodaira</u>

Sing.	Monodr.		ord(f)	ord(g)	ord(Δ)	Tate question
A_{n-1}	I_n	$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, $n \geq 1$	0	0	n	$\sqrt{a_2}\big\|_{s=0}$?
—	II	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	≥ 1	1	2	—
A_1	III	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	1	≥ 2	3	—
A_2	IV	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	≥ 2	2	4	$\sqrt{g/s^2}\big\|_{s=0}$
D_4	I_0^*	$\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$	≥ 2	≥ 3	6	$\circledast$
D_{4+n}	I_n^*, $n \geq 1$	$\begin{pmatrix} -1 & n \\ & -1 \end{pmatrix}$	2	3	$6+n$	$\circledast$

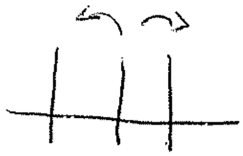
list continued

E_6	IV^*	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	≥ 3	4	8	$\sqrt{g/s^4}\Big\vert_{s=0}$
E_7	III^*	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	3	≥ 5	9	—
E_8	II^*	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	≥ 4	5	10	—
non-W minimal (++)			≥ 4	≥ 6		

(*) Take Q for I^*_{2k} : $\Delta / s^{2k+6}\Big|_{s=0}$

$$= \left(\frac{a_2}{s}\right)^2 \frac{b}{s^{2k+4}}$$

$$I^*_{2k-1} = \left(\frac{a_2}{s}\right)^3 \frac{b}{s^{2k+2}}$$

for D_4  . S_3 – monodromy

(+) j-fct.

	I_n	II	III	IV	I_0^*	I_n^*	II^*	IV^*	III^*
j	∞	0	1728	0	$a_0 b_0$ $\neq 0,1,\infty$	∞	0	0	[illegible]

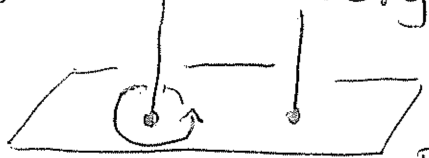
(++)

$$y^2 = x^3 + s^4 f_4 x + s^6 g_6$$

$$\left(\frac{y}{s^3}\right)^2 = \left(\frac{x}{s^2}\right)^3 + f_4\left(\frac{x}{s^2}\right) + g_6$$

$$\tilde{y}^2 = \tilde{x}^3 + f_4 \tilde{x} + g_6$$

Vafa's F-theory proposal:



B

monodr.

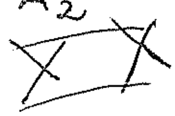
$n\,D_7$ branes : $SU(n)$ — A_{n-1} $\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}$

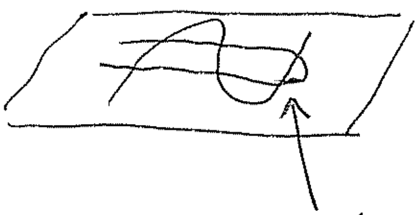
$n\,D_7$ branes + O_7 plane : $SO(2n)$, $\begin{pmatrix}-1 & n-4\\ 0 & -1\end{pmatrix}$ — D_n

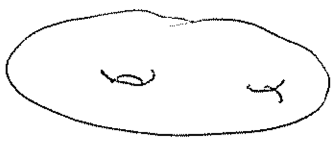
$$\left\{ \begin{array}{l} -\ E_6, E_7, E_8 \text{ sing} \\ \quad\updownarrow \\ E_6, E_7, E_8 \text{ grps} \end{array} \right.$$

<u>dim $B > 1$</u>:

A_2







$\Sigma \subseteq \Delta \subseteq B$

$\mathrm{ord}_\Sigma(f) \quad \mathrm{ord}_\Sigma(g) \quad \mathrm{ord}_\Sigma(\Delta)$

$\Rightarrow$ what is the identity of the general sing of fibre along that component dim B > 1

Kodaira	Tate	group
I_n, $n \geq 1$	Exists	$SU(n)$
	$\nexists$	$Sp([\frac{n}{2}])$
IV	$\exists$	$SU(3)$
	$\nexists$	$Sp(1)$
I_0^*	3 roots	G_2
	1 root	$SO(7)$
	0 roots	$SO(8)$
I_n^*	$\exists$	$SO(2n+8)$
	$\nexists$	$SO(2n+7)$
IV^*	$\exists$	E_6
	$\nexists$	F_4