Morrison: "Contractible Divisors in F-theory"

$$\begin{array}{c} \overline{X} \\ \downarrow \\ B \end{array}$$

elliptic fibration

Weierstraß: $y^2 = x^3 + f(b)\,x + g(b)$

$$f(b) \in H^0(\mathcal{L}^{\otimes 4})$$
$$g(b) \in H^0(\mathcal{L}^{\otimes 6})$$

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \overline{X} \arrow[d] \\ & B \end{tikzcd}$$

$$\begin{array}{c} X \\ \downarrow \\ B \end{array}$$

map of fiber dim 1

X nonsingular

dim $X = 1$ then we get a resolution provided that the singularities of $\overline{X}$ are "canonical"

Cases in which $\overline{X}$ has worse than canonical singularities, arising from

$$\begin{tikzcd} \tilde{\overline{X}} \arrow[r] \arrow[d] & \overline{X} \arrow[d] \\ E \subseteq \tilde{B} \arrow[r] & B \supseteq C \end{tikzcd}$$

$\dim C < \dim E$

$(B \setminus E) \cong (B \setminus C)$

Phenomenology applications: $\dim \tilde{B} = 3$, $E \subseteq \tilde{B}$ divisor supports the GUT gauge grp. G

(i.e., fibres of $\tilde{\overline{X}} \to \tilde{B}$ have Kodaira type along $\equiv$ corresponding to G)

<u>Kodaira table</u> $\Delta = 4f^3 + 27g^2$

$D \subset B$

$\nu_D(f)$	$\nu_D(g)$	$\nu_D(\Delta)$	Kodaira fiber
$\vdots$			
≥ 4	≥ 6	≥ 12	non-minimal

- Let $w = 0$ be local eqn. of D

$w^4 \mid f, \quad w^6 \mid g \implies w^{12} \mid \Delta$

$y^2 = x^3 + w^4 f_0(b) x + w^6 g_0(b)$

$\left(\frac{y}{w^3}\right)^2 = \left(\frac{x}{w^2}\right)^3 + f_0 \left(\frac{x}{w^2}\right) + g_0$

$Y = \frac{y}{w^3}, \quad X = \frac{x}{w^2}, \quad \mathcal{L}' = \mathcal{L} \otimes \mathcal{O}(-D)$

- if $\dim B \geq 2$, can ask at each point $p \in B$, where in the Kodaira table are we using

$\mathrm{mult}_p(f), \mathrm{mult}_p(g), \mathrm{mult}_p(\Delta)$?

The answer determines the <u>matter content</u> (i.e., determines a rep of G)

What is the meaning of being in non-minimal locus?
$\bar{X}$ has worse singularities, resolution has non-equidimensional fibers.

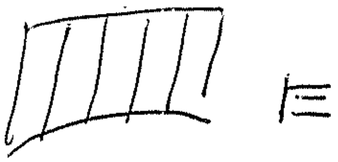
<u>Example</u>: $\dim B = 2$

$$\begin{array}{ccc} p & \in & B \\ \uparrow & & \uparrow \\ E & \subseteq & \tilde{B} = Bl_p B \end{array}$$

$p \in B$ in non-minimal locus

E = exc. curve
$E^2 = -1, \quad K_{\tilde{B}} \cdot E = -1$
$E \cong \mathbb{P}^1$

Get a Weierstraß model over $\tilde{B}$ by pull back.

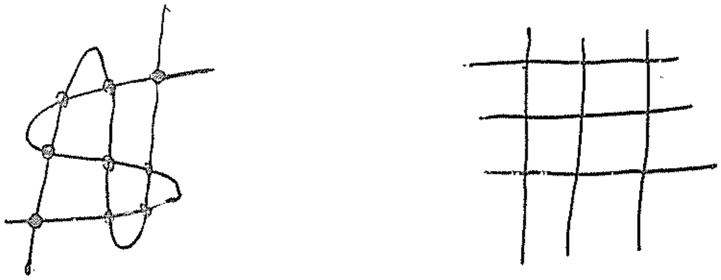


elliptic fibration over E
$\Rightarrow \mathcal{L}_E = \mathcal{O}(1)$

$$y^2 = x^3 + f_4(s,t)\, x + g_6(s,t)$$

homog in $s,t \leftrightarrow \mathbb{P}^1$

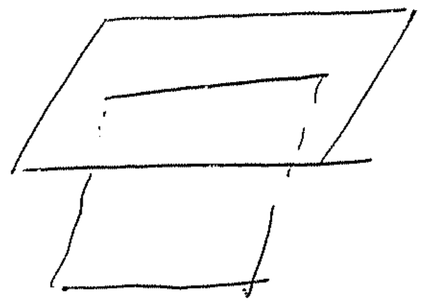
X_E = blowup of $\mathbb{P}^2$ in 9 pts



rational elliptic surface
dP_9 or $\frac{1}{2} K3$

$$\begin{tikzcd}
? \arrow[d] & & & \\
\overline{X} \arrow[dr] & \overline{\tilde{X}} \arrow[l] \arrow[d, dashed] \arrow[r] & \tilde{B} \arrow[dl] & E \arrow[l, phantom, "\supset"] \\
& B & &
\end{tikzcd}$$

$$\tilde{X} \to \overline{\tilde{X}}, \quad \tilde{X} \to \tilde{B}, \quad \tilde{X} \supset \tilde{S} = \text{RES (rational ell. surf)} \to E$$



$\sigma \subseteq \tilde{S}$

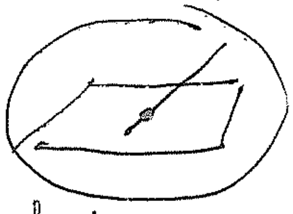
$N_{\sigma/\tilde{X}} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$

flop of $\tilde{X}$ along σ

$\mathbb{P}(\mathcal{O}(-1) \oplus \mathcal{O}(-$
$\|$
$\mathbb{P}^1 \times \mathbb{P}^1$

$\bar{\sigma}$

image of S



$\subset \overline{X} =$ ~~[illegible]~~

$S = Bl_8 \mathbb{P}^2 = dP_8$

$\hat{X}$ = resolution of $\overline{X}$

$\hat{\sigma} \subset X \supset \hat{S}$

$$
\begin{tikzcd}
X \arrow[d] \arrow[dd, bend right] & \tilde{X} \arrow[l, dashed] \arrow[d] & \tilde{X}_E = S \arrow[l, phantom, "\supseteq"] \arrow[d] \\
\bar{X} \arrow[d] & \bar{\tilde{X}} \arrow[l] \arrow[d] & \bar{\tilde{X}}_E \arrow[l, phantom, "\supset"] \arrow[d] \\
B & \tilde{B} \arrow[l] & E \arrow[l, phantom, "\supset"]
\end{tikzcd}
$$

rat. ell. surf

flop along (section) $\cap S$

Contractible divisors when $\dim B = 2$

$$
\begin{tikzcd}
E \arrow[d] \arrow[r, phantom, "\subseteq"] & \tilde{B} \arrow[d] \\
p \arrow[r, phantom, "\in"] & B
\end{tikzcd}
$$

First prop. E is contractible in AG $\iff E^2 < 0$

(contraction to alg space?)

Ass. $\mathcal{L} = \mathcal{O}_B(-K_{\tilde{B}})$ B non-singular
$\iff E^2 = -1$, $K_B \cdot E = -1$
$\implies E \cong \mathbb{P}^1$

Important fact about $\tilde{B}$:

$H^0(\mathcal{O}(-4K_{\tilde{B}}))$, $H^0(\mathcal{O}(-6K_{\tilde{B}}))$, $H^0(\mathcal{O}(-12K_{\tilde{B}}))$
are non-empty

$(-4K_{\tilde{B}}) \cdot \tilde{E} < 0$ then E is a comp. of $-4K_{\tilde{B}}$

$$K_{\tilde{B}} \cdot E + E^2 = 2g - 2 \geq -2$$
$$K_{\tilde{B}} \cdot E > -2$$
$$0 \leq -K_{\tilde{B}} \cdot E < 2$$

$\Rightarrow$ either $E^2 = -1$, $K_{\tilde{B}} \cdot E = -1$
$E \cong \mathbb{P}^1$

or $E^2 = -2$, $K_{\tilde{B}} \cdot E = 0$
$E \cong \mathbb{P}^1$.

In any other case,
E is a component of Δ
$\Rightarrow$ E supports some factor of the F-theory gauge grp

smallest m_4, m_6, m_{12}

$$(-4K_{\tilde{B}} - m_4 E) \cdot E \geq 0, \text{ etc.}$$

If we also insist that Weierstraß mode on $\tilde{B}$ is minimal, then $m_4 < 4$ or $m_6 < 6$

$\Rightarrow$ $E^2 \geq -12$

<u>Case: $E^2 = -2$, $K_{\tilde{B}} \cdot E = 0$</u>

$$\begin{array}{ccc} \tilde{B} & \supset & E \\ \downarrow & & \downarrow \\ B & \ni & p \end{array}$$

p is a singular pt of B locally iso to $\mathbb{C}^2/\mathbb{Z}_2$

$\tilde{X}_E$, $\mathcal{L}_E = 0$ $\quad K_{\tilde{B}} \cdot E = 0$

$\Rightarrow \tilde{X}_E \cong C_\tau \times E$

$B = B_0$, B_t is non-singular

In all cases other than $E^2 = -1$,
$K_{\tilde{B}} \cdot E = -1$,

B singular
$\frac{1}{12} \Delta = \Sigma a_i C_i \quad a_i \in \mathbb{Q}$

$K_B + \frac{1}{12} \Delta$ is numerically zero.

$(B, \Sigma a_i C_i)$
- singularities are not too bad („log terminal")
- $K_B + \Sigma a_i C_i$ numer. zero