On the other side of Pfaffian

Non abelian GLSM & non-complete intersection CYs.

★ Gauged Linear Sigma Model (Witten '93)

2D supersymmetric field theory $\xrightarrow{IR}$ SCFT.

G = compact group
$\rho : G \to GL(V)$ (in practice write a basis of V)
W = superpotential = G-invariant poly on V (coeffs are parameters)

Construct a quotient $V /\!/ G_{\mathbb{C}} := \mu^{-1}(r)/G$

Use moment map $\mu : V \to \mathrm{Lie}(G)^*$
$$\begin{array}{ccc} \mu : V & \longrightarrow & \mathrm{Lie}(G)^* \\ & \searrow & \downarrow \\ & & \mathrm{Lie}(Z(G))^* \end{array}$$
$Z(G)$ = center of G

Need to choose a value for μ, this value is a parameters. ← These are complexified by θ-angles

Physics: What are all supersymmetric vacua?
Ans: For certain ranges of parameters if G = abelian get CY variety.

Example 1: $V = \mathbb{C}^6$, basis $P, \Phi_1, \dots, \Phi_5$

$G = U(1)$ action: $-5, 1, \dots, 1$

$W = P F_5(\Phi)$

moment map: $\mu(P, \vec{\Phi}) = \frac{1}{2}\left(-5|P|^2 + |\Phi_1|^2 + \dots + |\Phi_5|^2\right)$

r = moment map value

$\underline{\underline{r > 0}}$: $\Rightarrow \vec{\Phi} \neq \vec{0}$

$\mathbb{P}^4 = \{ |\phi_1|^2 + \dots + |\phi_5|^2 = c \} / U(1)$

P is a coord in $\mathcal{O}_{\mathbb{P}^4}(-5)$

F-term conditions: $\frac{\partial W}{\partial P} = F$, $\frac{\partial W}{\partial \Phi_j} = P \frac{\partial F}{\partial \Phi_j}$

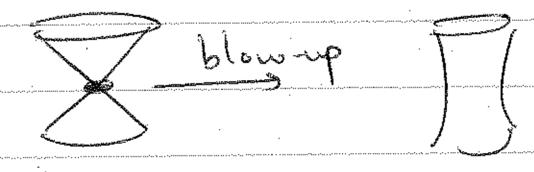
For F generic, $\frac{\partial F}{\partial \Phi_j} = 0 \; \forall j \Leftrightarrow \vec{\Phi} = \vec{0}$ excluded by D-term.

$P = 0, F = 0 \Leftarrow$

$\Rightarrow$ set of vacua is

$\{F = 0\} \subseteq \mathbb{P}^4 \to$ quintic CY

$\underline{r < 0}$: $P \neq 0$, using $U(1)$ action, $P = |p|$, residual $\mathbb{Z}_5$-action

$\mathbb{C}^5 / \mathbb{Z}_5$  $\to \mathcal{O}_{\mathbb{P}^4}(-5)$

F-term: $P \neq 0 \Rightarrow \frac{\partial F}{\partial \Phi_j} = 0 \; \forall j \Rightarrow \vec{\Phi} = \vec{0}$

$W_{eff} \propto F(\Phi)$ Landau-Ginzburg orbifold.

<u>General version</u>: G = compact abelian
$= U(1)^m \times \Gamma$

$$W = P_1 F_1(\Phi) + \dots + P_k F_k(\Phi)$$

$$\begin{array}{c} \\ U(1) \\ U(1) \\ \\ \end{array} \left(\begin{array}{c} P_1, \dots, P_k \; ; \; \Phi_1, \dots, \Phi_\ell \\ \\ \\ \\ \end{array} \right) \begin{array}{c} \\ \Sigma = 0 \\ \Sigma = 0 \\ \\ \end{array}$$

In the $r > 0$ phase, as before, we have the locus $\{F_1 = \dots = F_k = 0, \; P_1 = \dots = P_k = 0\}$ which is CY for appropriate parameters.

$[5] \subseteq \mathbb{P}^4$, $[2] \cap [4] \subseteq \mathbb{P}^5$,
$[3] \cap [3]$

$[2] \cap [2] \cap [3] \subseteq \mathbb{P}^6$ $\quad [2] \cap [2] \cap [2] \cap [2] \subseteq \mathbb{P}^7$

These are the only complete intersections in $\mathbb{P}^n$ which are CY.

Hypersurface $\subseteq \mathbb{P}^N$, nonsingular $\Rightarrow$ defined by 1 equation.

Codim 2 $\subseteq \mathbb{P}^N$, typical (Serre)

$$Y = X_1 \cap X_2$$
$$d_1 \quad d_2 \text{ - degrees}$$
$$F_1 = 0 \quad F_2 = 0$$

$$0 \to \mathcal{O}_{\mathbb{P}}(-d_1-d_2) \xrightarrow{[-F_2\ \ F_1]} \mathcal{O}_{\mathbb{P}}(-d_1)\oplus\mathcal{O}_{\mathbb{P}}(-d_2) \xrightarrow{\begin{bmatrix}F_1\\F_2\end{bmatrix}} \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_Y \to 0$$

$$\underbrace{[-F_2 \quad F_1]}_{1\times 2\ \text{matrix}} \begin{bmatrix}F_1\\F_2\end{bmatrix} = [0]$$

matrix of maximal minors of 1×2 matrix

Example of codim 3.

$$0 \to \mathcal{O}_{\mathbb{P}}(-d_1-d_2-d_3) \xrightarrow{[F_1\ F_2\ F_3]} \mathcal{O}_{\mathbb{P}}(-d_2-d_3)\oplus\mathcal{O}_{\mathbb{P}}(-d_3-d_1)\oplus\mathcal{O}_{\mathbb{P}}(-d_1-d_2) \xrightarrow{\begin{bmatrix}0 & F_3 & -F_2\\ -F_3 & 0 & F_1\\ +F_2 & -F_1 & 0\end{bmatrix}} \mathcal{O}_{\mathbb{P}}(-d_1)\oplus\mathcal{O}_{\mathbb{P}}(-d_2)\oplus\mathcal{O}_{\mathbb{P}}(-d_3) \xrightarrow{\begin{bmatrix}F_1\\F_2\\F_3\end{bmatrix}} \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_Y \to 0$$

<u>Buchsbaum-Eisenbud</u>: Assume hypothesis, codim 3, then

$$0 \to \mathcal{O}_{\mathbb{P}}(-t-2s) \xrightarrow{g^\vee(-t-2s)} \mathcal{E}^\vee(-t-s) \xrightarrow{f} \mathcal{E}(-s) \xrightarrow{g} \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_Y \to 0$$

f is skew-symm. $(2\gamma+1)\times(2\gamma+1)$

g = Pfaffian of $2\gamma\times 2\gamma$ matrix j^{th} row & col deleted.

Choose 7 skew-symm. 7×7 matrices

$A_1, \dots, A_7$: A_α

$A_\alpha^{ij} = -A_\alpha^{ji}$, A_α are "generic"

$p^1 A_1 + p^2 A_2 + \dots + p^7 A_7 =$ skew-sym 7×7
poly. entries

$\vec{p} \in \mathbb{C}^7 \quad \text{rank}(p^\alpha A_\alpha) \le 6.$

Generic assumption (*) Assume $\text{rank}(p^\alpha A_\alpha) < 4 \Rightarrow \vec{p} = \vec{0}$

$\{\vec{p} \in \mathbb{P}^6 \mid \text{rank}(p^\alpha A_\alpha) = 4\}$

$\shortparallel$

$\{\vec{p} \mid \text{Pf}((p^\alpha A_\alpha)_i) = 0 \ \forall \ i = 1, \dots, 7\}$

is called the Pfaffian subvariety of $\mathbb{P}^6$ defined by p^α, A_α.

Fact: it is a CY 3-fold.

p^α = homogeneous coords on toric variety X.

$A_\alpha = (2r+1)\times(2r+1)$ matrix

$\langle \text{Pf}((p^\alpha A_\alpha)_i) \rangle \subseteq X$.

GLSM: $G = U(1)^{m-1} \times \underbrace{U(2)}_{\text{or } U(1)\times SU(2)}$.

V has basis $p^1, \dots, p^N, (\Phi^a_j)_{a=1,2}$

Example: $G = U(2)$: $\epsilon_{ab} \Phi^a_j \Phi^b_k$ transforms like $\det^{-1}$ of $U(2)$.

$W = M \sum A^{jk}_\alpha p^\alpha \epsilon_{ab} \Phi^a_j \Phi^b_k$.

Vacua: $\mu(P, \Phi) = \frac{1}{2}\left(-\sum |P|^2 + \sum \Phi_j^\dagger \Phi_j\right)$

$r < 0$: $\vec{p} \neq \vec{0}$

D-term gives vector bundle / $SU(2)$
$$\downarrow$$
$$\mathbb{P}^6.$$

Hori/Tong ~~Fix a point~~ $\cancel{\vec{p} \in \mathbb{P}^6}$. ~~Consider~~ $\cancel{V_{\vec{p}} / SU(2)}$

F-term: $[A_\alpha P^\alpha]$ is a mass term for Φ^a_j

$\text{rank}(p^\alpha A_\alpha) = 2r \Rightarrow 1$ massless, $2r$ massive

$\text{rank}(p^\alpha A_\alpha) = 2r-2 \Rightarrow 3$ massless, $2r-2$ massive

In the first case, $rk(p^\alpha A_\alpha) = 2r$ there are no vacua. $(M \to \infty)$

$\text{rank}(P^\alpha A_\alpha) = 2r-2$. $M \to \infty$ physics $\to$ unique vacuum.