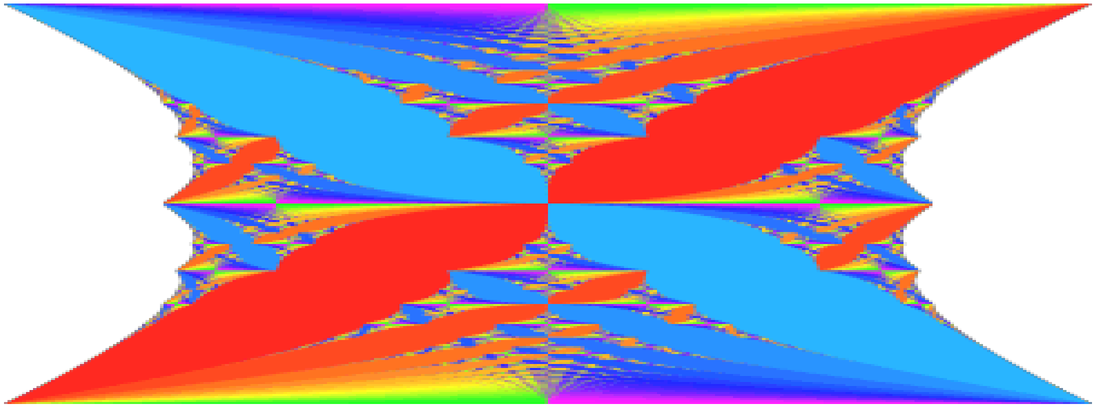
Topological Phases of Matter

Modeling and Classification



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Microsoft Station Q
RTG in Topology and Geometry, UCSB Oct 21, 2011

Prediction of Quantum Theory

- **Quantum computing is possible**

- **There are non-abelian anyons**

Thm: Prediction 2 implies Prediction 1.

Anyon=Localized Non-Local Properties

Favorite Theorems

- Poincare-Hopf Index Thm
- Gauss-Bonnet-Chern Thm

Quantum Systems

- A pair Q=($\mathcal{L}$, H), where $\mathcal{L}$ is a Hilbert space and H an Hermitian operator, physically H should be local.

- Examples:

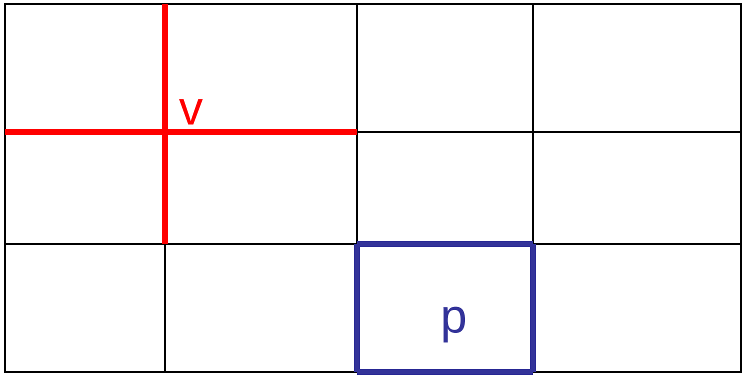
0) $\mathcal{L} = \otimes_i \mathbf{C}^2$, $H=\sum_i I \otimes \sigma_z{}^i \otimes I$, g.s.=$|1\rangle \otimes \ldots \otimes |1\rangle$, $\mathbf{C}^2 = \mathbf{C}\,|0\rangle \oplus \mathbf{C}\,|1\rangle$

1) Toric code---Z_2-homology (Turaev-Viro type TQFT or Levin-Wen model)

2) Hofstadter model---Chern number c_1 (Free fermions)

Toric Code

$$H=-g\sum_v A_v - J\sum_p B_p$$



$=T^2$

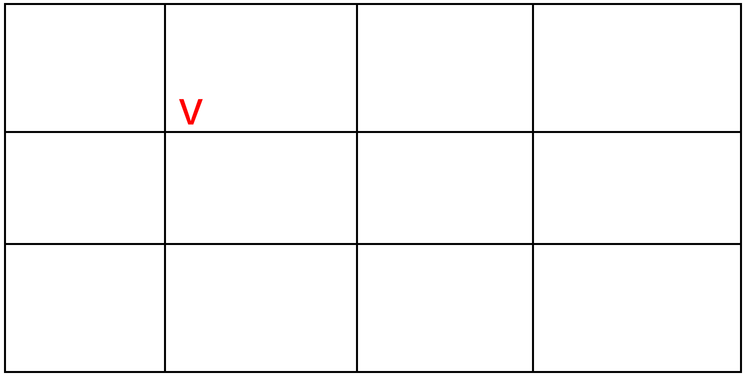
$$\mathcal{L} = \otimes_{edges} \mathbf{C}^2$$

$$A_v = \otimes_{e\varepsilon v} \sigma^z \otimes_{others} Id_e,$$

$$B_p = \otimes_{e\varepsilon p} \sigma^x \otimes_{others} Id_e,$$

Hofstadter Model

$$\mathbf{H}(\boldsymbol{\varphi}, \boldsymbol{\mu}) = -\sum_{\mathbf{v},\mathbf{v}'} \mathbf{h}_{\mathbf{v},\mathbf{v}'} \ \boldsymbol{a_v}^{+} \boldsymbol{a_{v'}} - \ \boldsymbol{\mu} \sum_{\mathbf{v}} \boldsymbol{a_v}^{+} \boldsymbol{a_v}$$



$=T^2$

Where $\boldsymbol{h}_{(\boldsymbol{m},\boldsymbol{n}),(\boldsymbol{m}',\boldsymbol{n}')}$

$\mathcal{L} = \otimes_{vertices}$ **C^2**

=1 if $\mathbf{m} = \mathbf{m}' \pm \mathbf{1}, \boldsymbol{n} = \boldsymbol{n}'$

=$\boldsymbol{e}^{\pm 2\pi i m \varphi}$ **if n=**$\boldsymbol{n}' \pm \mathbf{1}, \boldsymbol{m} = \boldsymbol{m}'$

=0 otherwise

and v=(m,n), v'=(m',n') are vertices, $\boldsymbol{a_v}^{+}$, $\boldsymbol{a_{v'}}$ **are fermion creation and annihilation operators at v, v'.**

All Physics Is Local

- A physical quantum system Q=($\mathcal{L}$, H) on a space Y has a decomposition $=\bigotimes_{\alpha} L_{\alpha}$ or $\bigoplus_{\alpha} L_{\alpha}$, and H is local w.r.t. the decomposition.

- An n-dim quantum theory is a Hamiltonian schema that defines a quantum system one each n-manifold (space) Y.

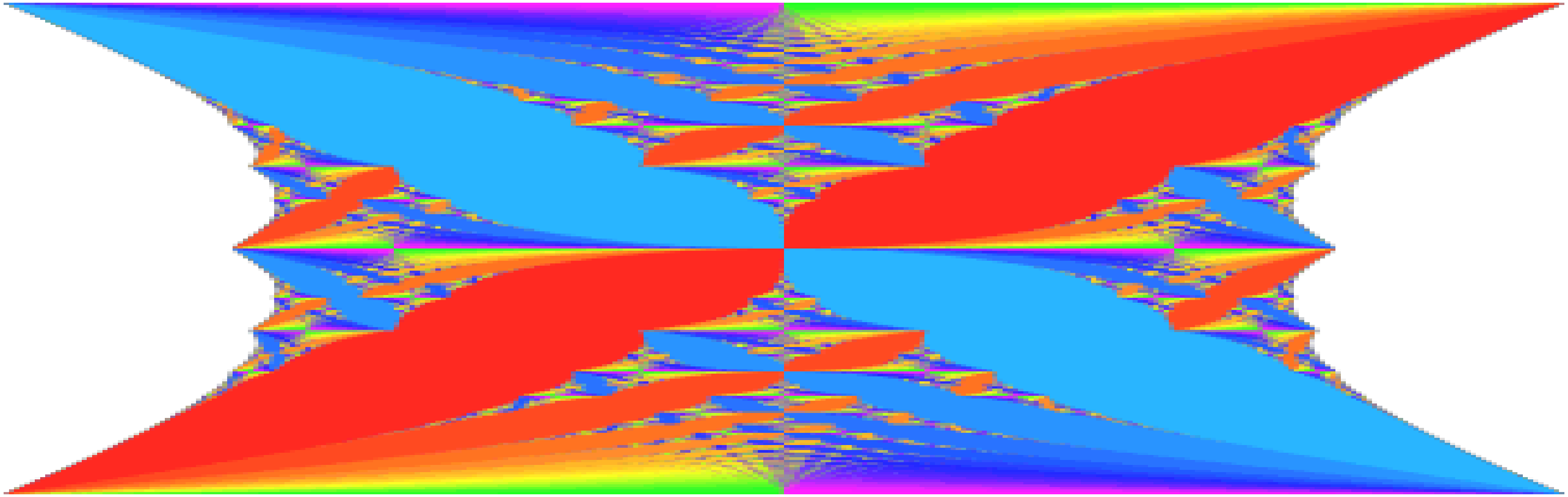
Phase Diagram

- Given a set of quantum systems Q(x) indexed by a parameter set X, a subset X\C of admissible ones, and an equivalence relation on X\C, then each **equivalence class** of X\C is a **phase**.

- The set X\C divided into phases is a phase diagram.

Hofstadter Butterfly

Fractal phase diagram of the Hofstadter model



Each of the infinite phases is characterized by the Chern number of its Hall conductance. Warm colors indicate positive Chern numbers; cool colors, negative numbers, and white region Chern numbers=0

H-axis=chemical potential, V-axis=magnetic flux

D. Osadchy, J. Avron, J. Math. Phys. 42, 2001

Topological Phases of Matter

A topological quantum phase is represented by a quantum theory whose low energy physics in the thermodynamic limit is modeled by a stable unitary topological quantum field theory (TQFT) and topological responses.

Remarks:

1. Low energy physics might be modeled only partially
2. Stability is related to energy gap

Ground States Form TQFTs

Given a quantum theory H on a physical space Y with Hilbert space $L_Y \cong \bigoplus V_i(Y)$, where $V_i(Y)$ has energy λ_i, and $V_0(Y)$ is the ground state manifold. If H is topological, then the functor $Y \rightarrow V(Y)$ is a part of a TQFT.

Classification of topological phases of matter, to first approximation, is to classify unitary topological quantum field theories?

Atiyah's Axioms of (n+1)-TQFT
(TQFT w/o excitations and anomaly)

A symmetric monoidal functor (V,Z):

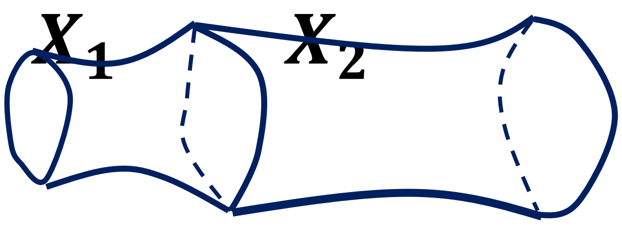
Bord(n+1)→ Vec

e.g. n=2, V(Y)=**C**[H_1(Y;Z_2)]

Oriented closed n-mfd Y → vector space V(Y)

Orient (n+1)-mfd X with ∂X=Y → vector Z(X)∈V(∂X)

- **V(∅) ≅ C**
- **$V(Y_1 \cup Y_2) \cong V(Y_1)\otimes V(Y_2)$**
- **$V(-Y) \cong V^*(Y)$**
- **$Z(Y\times I)=Id_{V(Y)}$**
- **$Z(X_1\cup_Y X_2)=Z(X_1)\bullet Z(X_2)$**



$Z(X_1)$ $Z(X_2)$

2D Topological Phases in Nature

- **Quantum Hall States**

  1980 Integral Quantum Hall Effect (QHE)---von Klitzing

  (1985 Nobel, **now called Chern Insulators**)

  1982 Fractional QHE---Stormer, Tsui, Gossard at v=1/3

  (1998 Nobel for Stormer, Tsui and Laughlin)

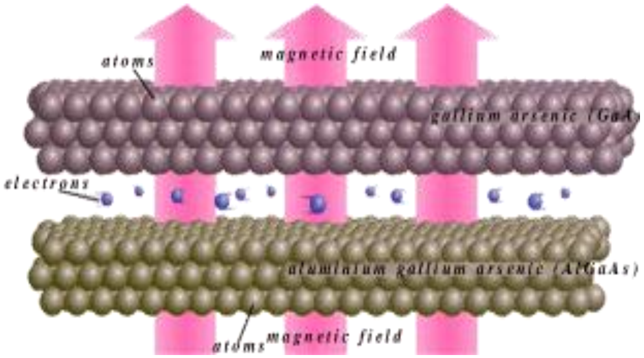
  1987 Non-abelian FQHE**???**---R. Willet et al at v=5/2

  (All are more or less Witten-Chern-Simons TQFTs)

- **Topological superconductor p+ip (Ising TQFT)**
- 2D topological insulator HgTe
- …

Quantum Hall States

N electrons in a plane bound to the interface between two semiconductors immersed in a perpendicular magnetic field



Classes of ground state wave functions that have similar properties or no phase transitions as N→∞ (N ~ 10^{11} cm^{-2})

Interaction is dynamical entanglement and quantum order is materialized entanglement

Fundamental Hamiltonian:

$$H = \Sigma_1{}^N \{\frac{1}{2m} [\nabla_j - q\, A(z_j)]^2 + V_{bg}(z_j)\} + \Sigma_{j<k} V(z_j - z_k)$$

Model Hamiltonian:

$$H = \Sigma_1{}^N \{\frac{1}{2m} [\nabla_j - q\, A(z_j)]^2\} + \textbf{?},\ \text{e.g. } \Sigma_{j<k}\, \delta(z_j - z_k)$$

z_j position of j-th electron

Classical Hall effect

On a new action of the magnet on electric currents

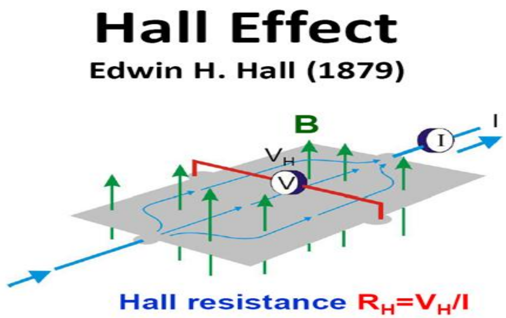
Am. J. Math. Vol. 2, No. 3, 287—292

E. H. Hall, 1879

“It must be carefully remembered, that the mechanical force which urges a conductor carrying a current across the lines of magnetic force, acts, not on the electric current, but on the conductor which carries it…”

Maxwell, Electricity and Magnetism Vol. II, p.144

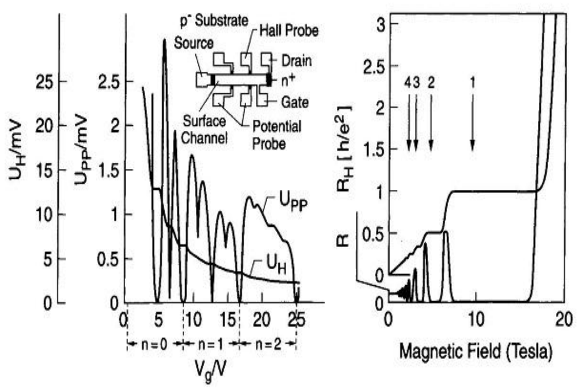
Birth of Integer Quantum Hall Effect



New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance,

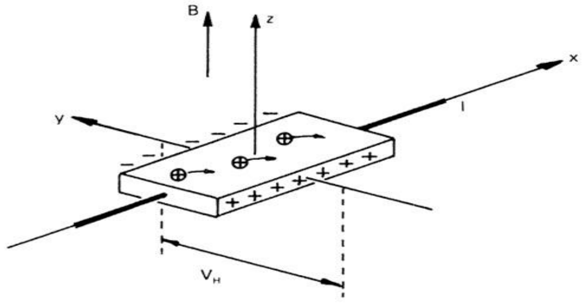
K. v. Klitzing, G. Dorda and M. Pepper

Phys. Rev. Lett. 45, 494 (1980).



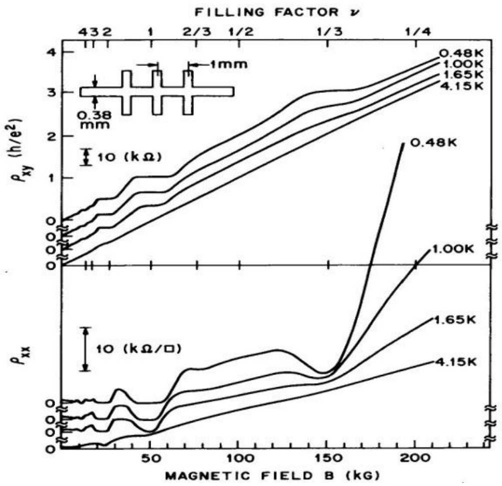
These experimental data, available to the public 3 years before the discovery of the quantum Hall effect, contain already all information of this new quantum effect so that everyone had the chance to make a discovery that led to the Nobel Prize in Physics 1985. The unexpected finding in the night of 4./5.2.1980 was the fact, that the plateau values in the Hall resistance x-y are not influenced by the amount of localized electrons and can be expressed with high precision by the equation $R_H = \frac{h}{\nu e^2}$

Fractional Quantum Hall Effect



D. Tsui enclosed the distance between B=0 and the position of the last IQHE between two fingers of one hand and measured the position of the new feature in this unit. He determined it to be three and exclaimed, "quarks!" H. Stormer

The FQHE is fascinating for a long list of reasons, but it is important, in my view, primarily for one: It established experimentally that both particles carrying an exact fraction of the electron charge e and powerful gauge forces between these particles, two central postulates of the standard model of elementary particles, can arise spontaneously as emergent phenomena. R. Laughlin

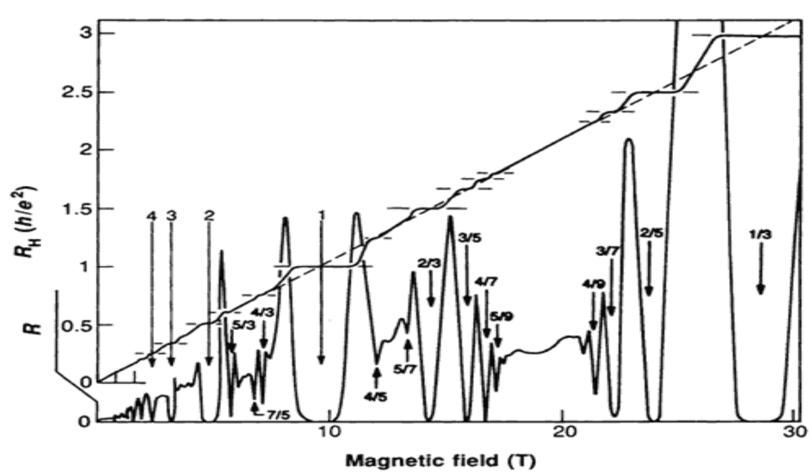


In 1998, Laughlin, Stormer, and Tsui are awarded the Nobel Prize

" for their discovery of a new form of quantum fluid with fractionally charged excitations."

D. C. Tsui, H. L. Stormer, and A. C. Gossard
Phys. Rev. Lett. 48, 1559 (1982)

How Many Fractions Have Been Observed? ~80



$\nu = \frac{N_e}{N_\phi}$ filling factor or fraction

N_e = # of electrons

N_ϕ =# of flux quanta

How to model the quantum state(s) at a filling fraction?

What are the electrons doing at a plateau?

1/3	1/5	1/7	1/9	2/11	2/13	2/15	2/17	3/19	5/21	6/23	6/25
2/3	2/5	2/7	2/9	3/11	3/13	4/15	3/17	4/19	10/21		
4/3	3/5	3/7	4/9	4/11	4/13	7/15	4/17	5/19			
5/3	4/5	4/7	5/9	5/11	5/13	8/15	5/17	9/19			
7/3	6/5	5/7	7/9	6/11	6/13	11/15	6/17	10/19			
8/3	7/5	9/7	11/9	7/11	7/13	22/15	8/17				
	8/5	10/7	13/9	8/11	10/13	23/15	9/17				
	11/5	12/7	25/9	16/11	20/13						
	12/5	16/7		17/11							
		19/7									

m/5, m=14,16, 19

5/2

7/2

19/8

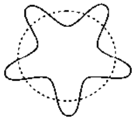
Pan et al (2008)


FEATURING
THE QUANTUM
SPINNERS
T = 0.05 K
ELECTRON
SQUARE
DANCE
TONIGHT
IN
CRYSTAL'S
BARN
PHYSICISTS
NOT
WELCOME

Pattern of long-ranged entanglement

All electrons participate in a collective dance following strict rules to form a non-local, internal, dynamical pattern---topological order

1. Electrons stay away from each other as much as possible

2. Every electron is in its own constant cyclotron motion 

3. Each electron takes an integer number of steps to go around another electron

ν=1/3

$$\psi_{1/3} = \prod_{i<j} (z_i - z_j)^3 e^{-\sum z_i \bar{z}_i /4}$$

R. Laughlin

U(1)-WCS theory, abelian anyons

ν=5/2 ?

$$\psi_{5/2} = Pf\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j)^2 e^{-\sum z_i \bar{z}_j /4}$$

Moore-Read

Ising TQFT or "SU(2)$_2$" WCS theory, non-abelian anyons

Classify Fractional Quantum Hall States

Wave functions of bosonic FQH liquids

- Chirality:

  $\Psi(z_1,\ldots,z_N)$ is a polynomial (Ignore Gaussian)
- Statistics:

  symmetric=anti-symmetric divided by $\Pi_{i<j}(z_i-z_j)$
- Translation invariant:

  $\Psi(z_1+c,\ldots,z_N+c) = \Psi(z_1,\ldots,z_N)$ for any c
- Filling fraction:

  $\nu=\lim \frac{N}{N_\phi}$, where N_ϕ is max degree of any z_i

Conformal blocks of CFTs→ TQFTs

FQH States =WCS TQFTs?

Physical Thm: Topological properties of abelian bosonic FQH liquids are modeled by Witten-Chern-Simons theories with abelian gauge groups T^n.

Conjecture: Topological properties of FQH liquids at $\nu=2+\frac{k}{k+2}$ are modeled (partially) by $SU(2)_k$-WCS theories. k=1,2,3,4, $\nu=\frac{7}{3}, \frac{5}{2}, \frac{13}{5}, \frac{8}{3}$. (Read-Rezayi). 5/2 √ physically

Expansion of Quantum Hall Physics

- Topological phases of free fermion systems—local gapped free fermions

- Topological phases with anyons in 2D---Schwartz type (2+1)-TQFTs including Witten-Chern-Simons theories

- Short-ranged entangled phases---Witten type cohomological TQFTs?

I: Free Fermions

$$H_h = \sum_{j,k} h_{j,k} \; a_j^+ a_k$$

h=($h_{j,k}$) is an lxl Hermitian matrix

Introduce Majorana operators

$$H_X = \frac{i}{4} \sum_{j,k} x_{j,k} \; \gamma_j^+ \gamma_k$$

X=($x_{j,k}$) is a real 2lx2l anti-symmetric matrix

“Gapped” , and “local”: the hopping matrix local $x_{j,k}$ =0 if |j-k| large.

Kitaev Periodic Table

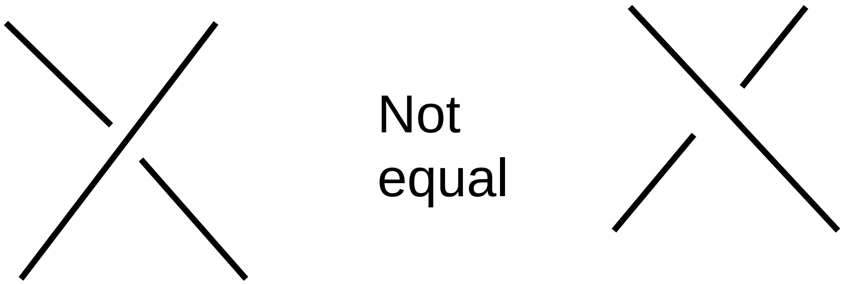
Symm\Dim d	0	1	2	3
Q	ZxU/UxU	U	ZxU/UxU	U
Q+SLS	U	ZxU/UxU	U	ZxU/UxU
No or P.H.S	O/U	O	ZxO/OxO	U/O
T only	U/Sp	O/U	O	ZxO/OxO
T and Q	Sp/SpxSpxZ	U/Sp	O/U	O
Three -1	Sp	Sp/SpxSpxZ	U/Sp	O/U
Four	Sp/U	Sp	Sp/SpxSp xZ	U/Sp
Five	U/O	Sp/U	Sp	Sp/SpxSp xZ
Six	Zx O/OxO	U/O	Sp/U	Sp
Seven	O	Zx O/OxO	U/O	Sp/U

Topological Invariants π_0

Symm\Dim d	0	1	2	3
A	Z	0	Z IQHE	0
AIII	0	Z	0	Z
D	Z_2	Z_2	Z	0
DIII	0	Z_2	Z_2 HgTe	Z
AII	Z	0	Z_2	Z_2
CII	0	Z	0	Z_2
C	0	0	Z	0
CI	0	0	0	Z
AI	Z	0	0	0
BDI	Z_2	Z	0	0

II: 2D with Anyons

In R^2, an exchange is of infinite order



Braids form groups B_n, then braid statistics of anyons is $\lambda: B_n \rightarrow U(k)$

If k=1, but not 1 or -1, abelian anyons

If k>1, but not in U(1), non-abelian

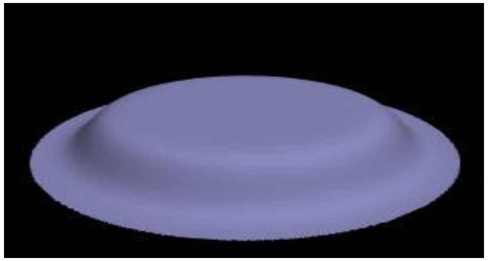
Laughlin wave function for ν=1/3

Laughlin 1983

Good trial wavefunction for N electrons at z_i in ground state

Gaussian

$$\Psi_{1/3} = \prod_{i<j} (z_i - z_j)^3 \, e^{-\Sigma_i |z_i|^2/4}$$



Physical Theorem:

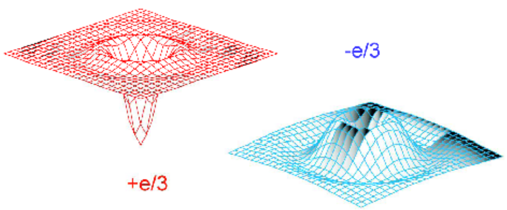
1. Laughlin state is incompressible: density and gap in limit (Laughlin 83)
2. Elementary excitations have charge e/3 (Laughlin 83)
3. Elementary excitations are abelian anyons (Arovas-Schrieffer-Wilczek 84)

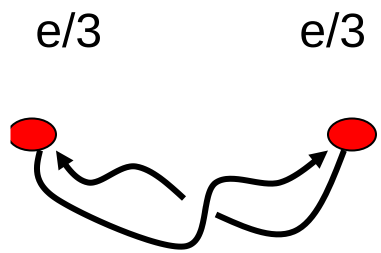
Experimental Confirmation:

1. and 2. √ , but 3. ?, thus Laughlin wave function is a good model

Elementary Excitations=Anyons

Quasi-holes/particles in ν=1/3 are abelian anyons





$$\Psi_{1/3} = \prod_k (\eta_0 - z_j)^3 \prod_{i<j} (z_i - z_j)^3 \; e^{-\Sigma_i |z_i|^2/4}$$

$$= \prod_k (\eta_1 - z_j) \prod_k (\eta_2 - z_j) \; \prod_k (\eta_3 - z_j) \prod_{i<j} (z_i - z_j)^3 \; e^{-\Sigma_i |z_i|^2/4}$$

n anyons at well-separated η_i, i=1,2,.., n,
there is a **unique** ground state

$$\psi \rightarrow e^{\pi\, i/3}\, \psi$$

Non-abelian Anyons

Given n anyons of type x in a disk D, their ground state degeneracy

dim(V(D,x,…,x))=$D_n \sim d^n$

The asymptotic growth rate d is called the quantum dimension.

An anyon d=1 is called an abelian anyon, e.g. Laughlin anyon, d=1

An anyon with d >1 is an non-abelian anyon, e.g. the Ising anyon σ, d=$\sqrt{2}$.

For n even, $D_n = \frac{1}{2}\, 2^{\frac{n}{2}}$ with fixed boundary conditions,

n odd, $D_n = 2^{\frac{n-1}{2}}$. (Nayak-Wilczek 96)

Degeneracy for non-abelian anyons in a disk grows exponentially with # of anyons, while for an abelian anyon, no degeneracy---it is always 1.

Non-abelian Statistics

If the ground state is not unique, and has a basis $\psi_1, \psi_2, \ldots, \psi_k$

Then after braiding some particles:

$$\psi_1 \longrightarrow a_{11}\psi_1 + a_{12}\psi_2 + \ldots + a_{k1}\psi_k$$

$$\psi_2 \longrightarrow a_{12}\psi_1 + a_{22}\psi_2 + \ldots + a_{k2}\psi_k$$

....

$$\lambda: B_n \longrightarrow U(k),$$

when k>1, non-abelian anyons.

Moore-Read or Pfaffian State

G. Moore, N. Read 1991

Pfaffian wave function (MR w/ ≈ charge sector)

$$\Psi_{1/2} = \mathrm{Pf}(1/(z_i - z_j)) \prod_{i<j} (z_i - z_j)^2 \, e^{-\Sigma_i |z_i|^2/4}$$

Pfaffian of a 2n×2n anti-symmetric matrix M=(a_{ij}) is

ω^n =n! Pf (M) dx^1∧dx^2∧…∧dx^{2n} if $\omega = \Sigma_{i<j} a_{ij}$ dx^i∧ dx^j

Physical Theorem:

1. Pfaffian state is gapped
2. Elementary excitations are non-abelian anyons, called Ising anyon σ

…… Read 09

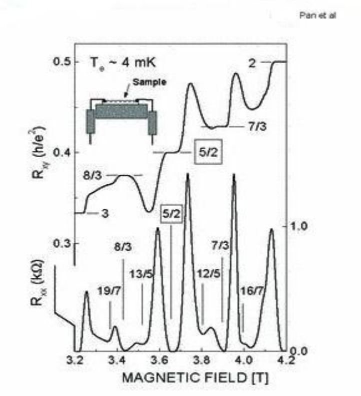
Enigma of ν=5/2 FQHE

R. Willett et al discovered ν=5/2 in1987

- Moore-Read State, Wen 1991
- Greiter-Wilczek-Wen 1991
- Nayak-Wilczek 1996
- Morf 1998
- …

MR (maybe some variation) is a good trial state for 5/2

- Bonderson, Gurarie, Nayak 2011,



Willett et al, PRL 59 1987

A landmark (physical) proof for the MR state

"Now we eagerly await the next great step: experimental confirmation." ---Wilczek

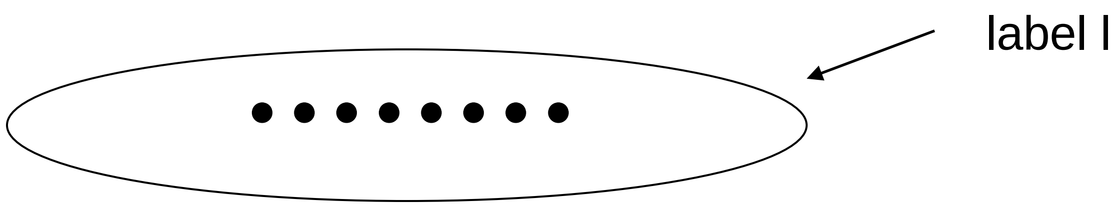
Experimental confirmation of 5/2:

gap and charge e/4 √ , but non-abelian anyons ???

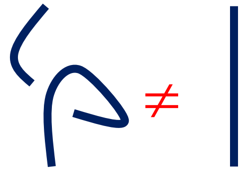
Extended (2+1)-TQFT

Put a theory H on a closed surface Y with anyons a_1, a_2, …, a_n at $\eta_1,\ldots,\eta_n$ (punctures), the (relative) ground states of the system "outside" $\eta_1,\ldots,\eta_n$ is a Hilbert space V(Y; a_1, a_2, …, a_n).

For anyons in a surface w/ boundaries (e.g. a disk), the boundaries need conditions.



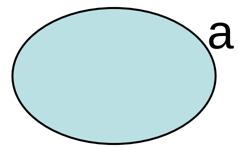
Stable boundary conditions correspond to anyon types (labels, super-selection sectors, topological charges). Moreover, each puncture (anyon) needs a tangent direction, so anyon is modeled by a small arrow (combed point), not just a point.



Extended (2+1)-TQFT Axioms

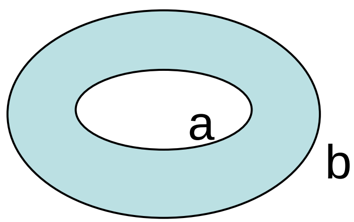
Moore-Seiberg, Walker, Turaev,…

Let L={a,b,c,…d} be the labels (particle types), a → a*, and a**=a,
0 (or 1) =trivial type



Disk Axiom:

$V(D^2; a)=0$ if $a \neq 0$, $\mathbb{C}$ if $a=0$

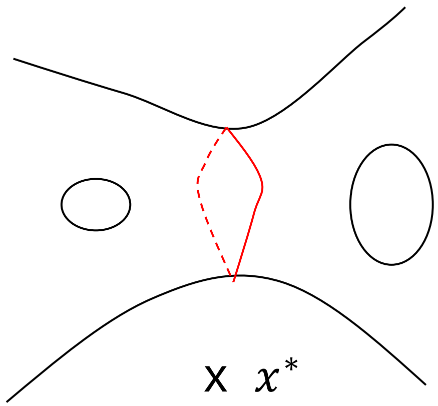


Annulus Axiom:

$V(A; a,b)=0$ if $a \neq b^*$, $\mathbb{C}$ if $a=b^*$

Gluing Axiom:

$V(Y; l) \cong \oplus_{x \varepsilon L} V(Y_{cut}; l, x, x^*)$



Algebraic Theory of Anyons

L={a,b,c,…d} a label set and $P_{ab,c}$ a pair of pants labeled by a,b,c.

$N_{ab,c}$=dim V($P_{ab,c}$), then $N_{ab,c}$ is the fusion rule of the theory.

a⊗b=$\oplus N_{ab,c}$c

Every surface Y can be cut into disks D, annuli A, and pairs of pants. If V(D), V(A), V($P_{ab,c}$) are known, then V(Y) is determined by the gluing axiom.

Conversely a TQFT can be constructed from V(Y) of disk, annulus and pair of pants. Need **consistent conditions: a modular tensor category**

Unitary modular categories are algebraic data of unitary (2+1)-TQFTs and algebraic theories of anyons: anyon=simple object, fusion=tensor product, statistics of anyons are representations of the mapping class groups.

Rank < 5 Unitary Modular Categories

joint work w/ E. Rowell and R. Stong

	A 1 Trivial			
	A 2 Semion		NA 2 Fib BU	
	A 2 (U(1),3)	NA 8 Ising	NA 2 (SO(3),5) BU	
A 5 Toric code	A 4 (U(1),4)	NA 4 Fib x Semion BU	NA 2 (SO(3),7) BU	NA 3 DFib BU

The ith-row is the classification of all rank=i unitary modular tensor categories. Middle symbol: fusion rule. Upper left corner: A=abelian theoy, NA=non-abelian. Upper right corner number=the number of distinct theories. Lower left corner BU=there is a universal braiding anyon.

Witt Group

- Two modular categories are Witt equivalence if they are the same up to Drinfeld centers

- All equivalence classes form an Abelian group.

III. Short-ranged Entangled

- Group cohomology

  X.-G. Wen et al

  Complete classification of 1D gapped phases

- Generalized cohomology theory

  A. Kitaev

Table of Topological Phases of Matter

Mathematically, define and classify unitary TQFTs

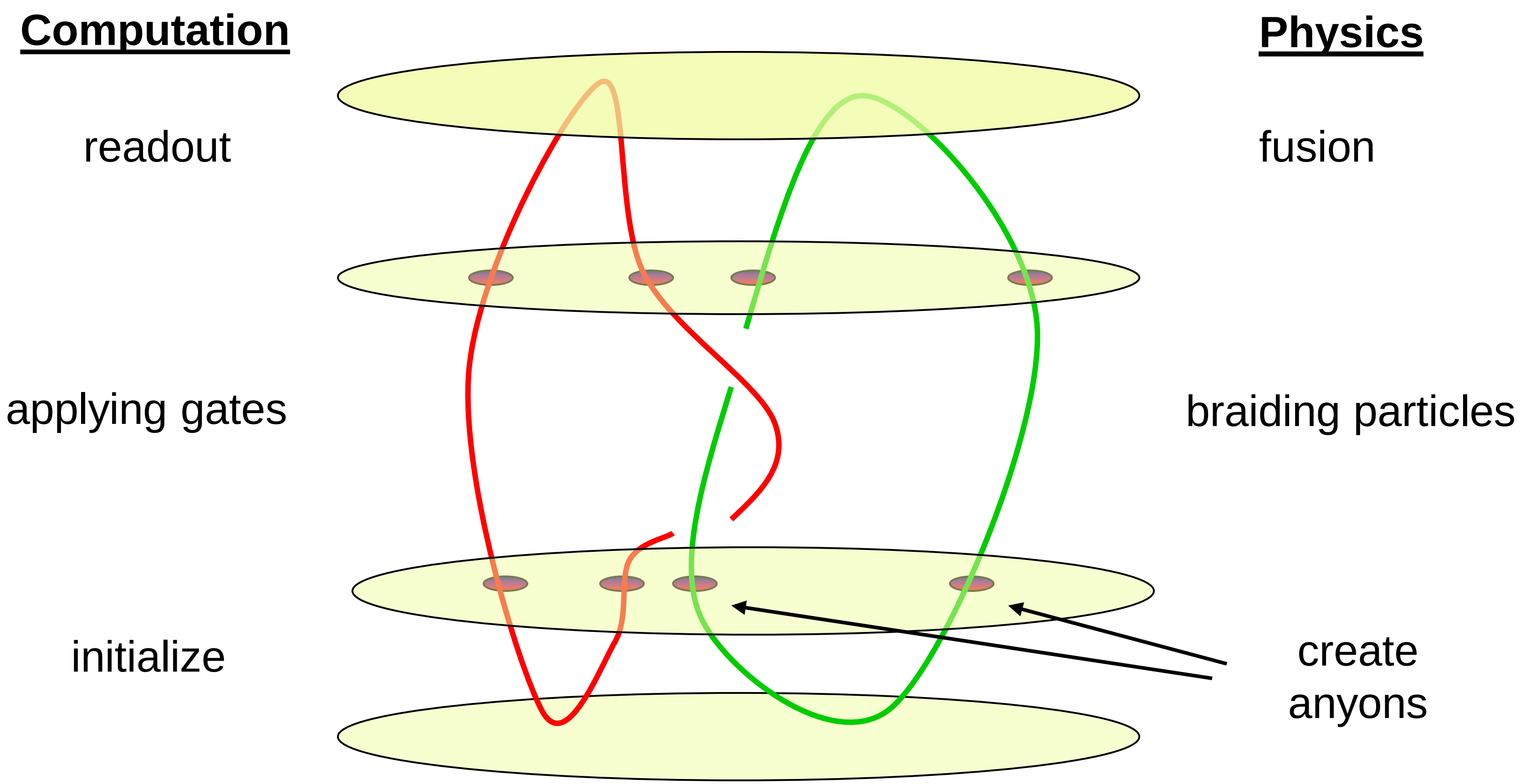
- Stability? Energy gap
- How to combine TQFTs with symmetry?
- Where is the geometry?

"All physics is geometry"---J. A. Wheeler

Quantum topology + Quantum geometry

to better understand quantum phases of matter

Topological Quantum Computation



Topological Quantum Computation