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Gapped Phases of Matter

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1) Gauge theory

$\Downarrow$

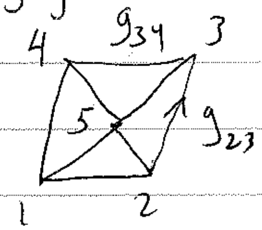
TQFT

2) Symmetry-protected topological phases of matter
(SPT phases)

$$H^D(BG; \mathbb{R}/\mathbb{Z})$$

$G \subset G_0$ (Lie group)
$\uparrow$
finite

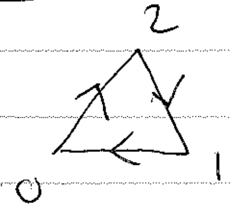
Dijkgraaf-Witten theory (purely Higgs)



$g_{23}, g_{34}, \ldots \in G$

$$Z = \sum_{g_{ij} \in G} f(g)$$

Constraint:



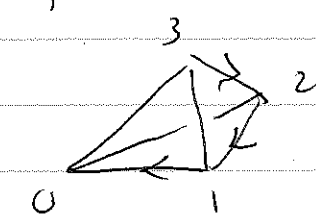
$g_{10}\, g_{21}\, g_{02} = 1$ for every triangle in the triangulation.

gauge transformation: $g_{10} \to h_0\, g_{10}\, h_1^{-1}$ etc where $h_0, h_1 \in G$.

$$f = \prod_{D\text{-simplices } p} f_p(g) \qquad (D = \text{top dimension})$$

$f_p(g) = f_p(g_1, \ldots, g_D)$ i.e., values on D edges

eg.



g_{10}, g_{21}, g_{32} are enough

$$f_p(g_1, \ldots, g_D) \in C^D(G, U(1)) \quad \text{s.t. } \delta f_p = 0.$$

1) Dijkgraaf-Witten:

action for topological gauge theory in D dim's are labelled by $\chi \in H^D(BG, U(1))$

$BG = K(G,1)$, $\pi_1(BG) = G$, $\pi_i(BG) = 0$, $i > 1$.

2) Confinement

$$G_0 \to G_{\text{low-energy}}$$

Confining TQFT

$$\Pi_2 \to H \xrightarrow{t} G_0$$

Π_2 is abelian

e.g. $\overset{\text{kernel}}{\underset{\text{(finite)}}{}}$ $SU(2)$ $SO(3)$

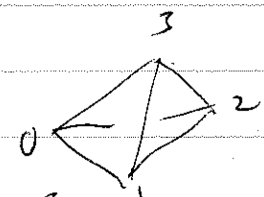
Π_2 = "magnetic gauge group" (preserve 't Hooft fluxes)

On lattice:

h associated to 2-simplex

$h \in \Pi_2$

on a 3 simplex,



$$\sum_{i=0}^{3} (-1)^i h_i = 0$$

with gauge invariance

$$Z = \sum_{\substack{h_p \\ p \text{ a 2-simplex}}} \text{weight}(h_p)$$

For a D-simplex, there are $\frac{D(D-1)}{2}$ variables h

Let's define a map to $B^2\Pi_2 = K(\Pi_2, 2)$

$$\pi_i(B^2\Pi_2) = 0 \text{ if } i \neq 2$$
$$\pi_2(B^2\Pi_2) = \Pi_2$$

Actions for a confining TQFT are labelled by
$x \in H^D(B^2\Pi_2, U(1))$

e.g. for $D=4$; see Eilenberg–MacLane (1947):

$$H^4(B^2\Pi_2, U(1)) = \{\text{quadratic functions on } \Pi_2 \text{ with values in } U(1)\}$$

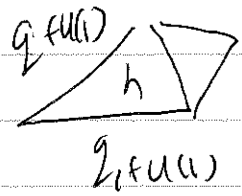
$\mathbb{Z}/N \to SU(N) \to SU(N)/\mathbb{Z}_N$

$$Q(\mathbb{Z}/N, U(1)) = \begin{cases} \mathbb{Z}/N, \ N \text{ odd} & f(x) = \frac{x^2}{N}p \\ \mathbb{Z}/2N, \ N \text{ even} & f(x) = \frac{x^2}{2N}p \end{cases}$$

$$S = \frac{1}{g^2}\int \mathrm{Tr}\, F \wedge * F + \frac{\theta}{8\pi^2}\int \mathrm{Tr}\, F \wedge F$$

$$F = dA + A \wedge A, \quad A \in \mathrm{Conn}(P, M)$$

$\theta = \frac{p}{N}$



$h \in \mathbb{Z}_N$

Common generalization:

Crossed module (replacing G or Π_2)

1) Instead of G, start with a "finite" weak 2-group.

2) (G, H, t, α)

G = group, H = group

$H \xrightarrow{t} G$ homomorphism

$\alpha : G \to \mathrm{Aut}(H)$

s.t. $t(\alpha_g(h)) = g\, t(h)\, g^{-1}$

$\alpha_{t(h)}(h') = h h' h^{-1}$

[Note: equivalence of 2-groups can change G and H]

3) $\Pi_2 \to H \xrightarrow{t} G \to \Pi_1$

Π_2 = kernel, Π_1 = cokernel

Can check: Π_2 is abelian, Π_1 is a group and Π_1 acts on Π_2 (via α)

$\mathbb{G} = (\Pi_1, \Pi_2, \alpha, \beta)$ ~~α action of~~

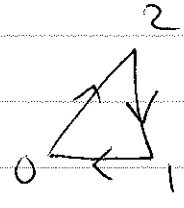
$\alpha : \Pi_1 \to \mathrm{Aut}(\Pi_2)$

$\beta \in H^3(B\Pi_1, \Pi_2)$

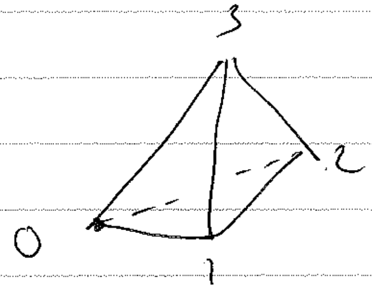
"2-stage Postnikov tower"

π_1 = unbroken electric gauge group.

π_2 = magnetic gauge group.



$g_{10}\, g_{21}\, g_{02} = 1$



$$\alpha_{g_{10}}(h_0) - h_1 + h_2 - h_3 = \beta(g_{10}, g_{21}, g_{32})$$

$$\begin{array}{ccc} B^2\pi_2 & \to & BF \\ & & \downarrow \\ & & B\pi_1 \end{array}$$

so not an Eilenberg-MacLane space

Actions labelled by $x \in H^D(B\mathcal{G}, U(1))$.

$D=2$ equiv to Dijkgraaf-Witten

$D=3$ gauged Sigma model

$\underline{D=4}$

Action is a sum of 3 pieces:

$$\int_M q(B) + \int g^*\lambda \cup B + \int_M g^*\omega$$

$$g : M \to B\Pi_1$$

$$\omega \in H^4(B\Pi_1, U(1))$$

$$\lambda \in H^2(B\Pi_2, \check{\Pi}_2), \quad \check{\Pi}_2 = \mathrm{Hom}(\Pi_2, U(1))$$

$q \in H^4(B^2\Pi_2, U(1))$ is a quadratic function