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A Malmendier

K3 surfaces from Seiberg-Witten curves

w/ Chuck Doran

Summary: We constructed multi-parameter families of K3 surfaces from all extremal rational elliptic surfaces, and computed period maps.

(1) Example Kummer pencil. Start with 2 elliptic curves

E_i, $i=1,2$ $\qquad y_i^2 = x_i(x_i-1)(x_i-\lambda_i)$

$$\lambda_i \notin \{0,1\} \qquad \lambda_i = \lambda_i(\tau_i) \leftarrow \text{Hauptmodul of } \Gamma(2)$$

$$J_i = j(\tau_i) = \frac{(\lambda_i^2-\lambda_i+1)^3}{\lambda_i^2(\lambda_i-1)^2}$$

$$A = E_1 \times E_2 \xrightarrow{\ \iota\ } A$$

$$(x_1,y_1,x_2,y_2) \longmapsto (x_1,-y_1,x_2,-y_2)$$

$$\mathrm{Kum}(E_1\times E_2) = \widehat{E_1\times E_2/\iota} = X$$

projection

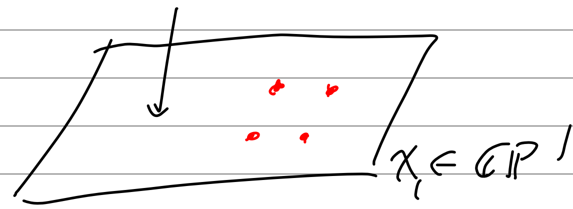
$$E_1 \xleftarrow{\ \pi_1\ } A \xrightarrow{\ \pi_2\ } E_2$$

induces an elliptic fibration on X

<u>inv. variables</u> $\quad x_1, \quad X = y_1^2 x_2, \quad Y = 2y_1^2(y_1 y_2)$

$$Y^2 = 4X(X - y_1^2)(X - y_1^2 \lambda_2)$$

functions of x_1



$$Y^2 = 4X^3 - g_2(x_1)X - g_3(x_1)$$

$$g_2 = \frac{4}{3}(\lambda_2^2 - \lambda_2 + 1)y_1^2$$

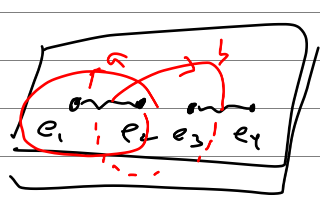
$$g_3 = \frac{4}{27}(\lambda_2 + 1)(\lambda_2 - 1)(2\lambda_2 - 1)y_1^3$$

$$\Delta = g_2^3 - 27g_3^2 = 16\lambda_2^2(\lambda_2 - 1)^2 y_1^6$$

Singular fibers at $x_1 = 0, 1, \lambda_1, \infty$ of type I_0^* ($\leadsto D_4$)

$$j = \frac{g_2^3}{\Delta} = j(\tau_2) \qquad \text{isotrivial}$$

fiber: holo 1-form $\frac{dx}{y}$ $\qquad \int_{a/b} \frac{dx}{y}$



$$\int_{a/b} \frac{dx}{Y} = \int_{a/b} \frac{dX}{2\sqrt{X(X-y_1^2)(X-y_1^2\lambda_2)}}$$

rescale $X \rightsquigarrow X y_1^2$

$$= \frac{1}{y_1} \int_{a/b} \frac{dx_2}{\sqrt{x_2(x_2-1)(x_2-\lambda_2)}} = \frac{1}{y_1} \begin{cases} 2\omega_2 \\ 2\omega_2' \end{cases}$$

with $\dfrac{2\omega_2'}{2\omega_2} = \tau_2$

Integrate again over base

$$\int_{a/b} dx_1 \, \frac{1}{y_1} \begin{cases} 2\omega_2 \\ 2\omega_2' \end{cases} = 4 \begin{cases} \omega_1 \omega_2 \\ \omega_1 \omega_2' \\ \omega_1' \omega_2 \\ \omega_1' \omega_2' \end{cases}$$

$$\omega_2 = {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1, \lambda_2\right)$$

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Solutions to rank 4 linear system.

Point of view of K3:

$$\int_{\text{base}} dx_1 \int_{\text{fibre}} \frac{dX}{Y} = \iint \underbrace{\overbrace{dx_1 \wedge \frac{dX}{Y}}^{?}}_{\text{Holo 2-form on K3}}$$

$$X_{\lambda_1, \lambda_2} \longrightarrow \left[\iint_{S_1} \eta, \iint_{S_2} \eta, \iint_{S_3} \eta, \iint_{S_4} \eta \right] \in \mathbb{P}^3.$$

$$NS(X) = H^{(1,1)}(X) \cap H^2(X, \mathbb{Z}) \qquad \text{rank} = \rho = 18$$

$$T(X) = NS(X)^{\perp}$$

$$\frac{\omega_2'}{\omega_2} = \tau_2, \quad \frac{\omega_1'}{\omega_1} = \tau_1 \quad \text{so} \quad \begin{aligned} x &= \omega_1 \omega_2 \\ y &= \omega_1 \omega_2' \\ z &= \omega_1' \omega_2 \\ w &= \omega_1' \omega_2' \end{aligned}$$

quadratic relation $\underbrace{xw - yz}_{Q(\vec{x}) \rightsquigarrow \beta(\vec{x}, \vec{y}) = Q(\vec{x}+\vec{y}) + \cdots} = 0$

$$Q(\overline{\vec{x}, \vec{x}}) > 0$$

[Continued on slides]