K3 surfaces from Seiberg-Witten curves

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F-theoretic interpretation of Kummer pencil (Sen '95)

- Kummer pencil on $T^4/\mathbb{Z}_2$ is 2dim subspace of moduli space:
  1) Base is $T^2/\mathbb{Z}_2$ with complex structure $\tau_b = \tau_1$.
  2) Elliptic fiber has constant modulus $\tau_f = \tau_2$.
  3) Over $z = 0, 1, \lambda_1, \infty$ there are I_0^*-fibers ($\text{ord}_D(\Delta) = 6$).
  4) Monodromy acts by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ on periods, but trivially on τ_f.
- Theory is equivalent to orientifold in type IIB string
  1) Base is $T^2/\mathbb{Z}_2$.
  2) Axion-dilaton modulus is constant.
  3) 4 sets of 6 coincident D7-branes located at $z = 0, 1, \lambda_1, \infty$
  4) Monodromy is identified with transformation $(-1)^{F_L} \cdot \Omega$

Embedding of SW-curve into F-theory

- Physics near orientifold plane = Seiberg-Witten solution for $\mathcal{N}=2$, $d=4$ SYM for $SU(2)$ with four quark flavors.
- Sen provided embedding of SW-curve into F-theory:

	$\mathcal{N}=2$ **SYM**	$\leftrightarrow$	**F-theory**
total space:	rational elliptic surface		elliptic K3 surface
elliptic fiber:	gauge coupling		axion-dilaton modulus
base:	vev u of adjoint scalar		base pt z of IIB compactification
limit:	all quarks massless	$\leftrightarrow$	orbifold limit $T^4/\mathbb{Z}_2$
fibration:	$2I_0^*$ (isotrivial)	$\leftrightarrow$	$4\ I_0^*$ (isotrivial)
parameter:	$j(\tau_0)$	$=$	$j(\tau_f)$
WEq:	$g_2(u) \doteq E_4(\tau_0)\,(u-1)^2$ $g_3(u) \doteq E_6(\tau_0)\,(u-1)^3$	$\leftrightarrow$	$G_2(z) = g_2(z)\,z^2(z-\lambda_1)^2$ $G_3(z) = g_3(z)\,z^3(z-\lambda_1)^3$
deformation:	masses $m_i > 0$	$=$	location of seven branes c_i
gen. fibration:	$6I_1 + I_0^*$	$\leftrightarrow$	$6I_1 + I_0^* + 2\ I_0^*$
WEq:	$g_2(u) = f_2(u, m_i)$ $g_3(u) = h_3(u, m_i)$	$\leftrightarrow$	$G_2(z) \quad = f_2(z, c_i)\,z^2(z-\lambda_1)^2$ $G_3(z) \quad = h_3(z, c_i)\,z^3(z-\lambda_1)^3$

Results (Doran, M.):

- replaced SW-curve ($2\,I_0^*$) with any extremal rational elliptic surfaces $\mathbf{S}$ with section (classified by **Miranda, Persson '86**),
- obtained 2-parameter families $\mathbf{X}_1(\mathbf{S})$ of lattice polarized K3 surfaces of Picard rank $\rho = 8 + 2 \cdot 4 + 2 = 18$, and 2-parameter families $\mathbf{X}_2(\mathbf{S})$ as their double covers,
- K3 periods satisfy system of linear PDEs of rank 4 (fibrewise periods of rational surface $\xrightarrow{\text{Euler tr.}}$ K3-periods),
- after restricting to simple one-parameter subfamily: reproduced periods for families of M_n-polarized K3 surfaces ($n = 1, 2, 3, 4, 6$), new family whose periods satisfy the Apery recurrence for $\zeta(2)$,
- determined quadratic period relation and interpretation of periods and twist-parameters as modular forms,
- generalization to 3-parameter family w/ $\rho = 17$ selects SW-curve ($N_f = 0$) $\rightarrow$ moduli: genus-2 curve of level 2

Rational surfaces

- Rational elliptic surfaces **S** over $\mathbb{CP}^1$ with section:

$$\bar{\mathbf{S}}: \; y^2 = 4\,x^3 - g_2\,x - g_3\,, \qquad \begin{array}{l} g_2 \in H^0(\mathcal{O}(4)), \\ g_3 \in H^0(\mathcal{O}(6)), \end{array} \quad [t:1] \in \mathbb{CP}^1.$$

- Consider extremal rational elliptic srfc: $\mathrm{rk}(\mathrm{MW}) = 0, \rho = 10.$

Examples:

- SW-curve **S** for pure $\mathrm{SU}(2)$-gauge theory: Legendre family over the t-line, t Hauptmodul for $\Gamma(2)$, $y^2 = x(x-1)(x-t)$

E_{sing}	I_2	I_2	$I_2^*(= D_6)$
t	0	1	∞

- Pencil related by 2-isogeny

E_{sing}	I_1	I_1	$I_4^*(= D_8)$
t	0	1	∞

Extremal rational surfaces and their periods

- Rational elliptic surfaces $\mathbf{S}$ with section

$$\bar{\mathbf{S}}: \ y^2 = 4\,x^3 - g_2\,x - g_3\,, \qquad \begin{array}{l} g_2 \in H^0(\mathcal{O}(4)), \\ g_3 \in H^0(\mathcal{O}(6)), \end{array} \quad [t:1] \in \mathbb{CP}^1.$$

- Extremal rational surfaces (up to $*$-transfer):

isotrivial			G
I_0	I_0^*	I_0^*	$\mathbb{Z}_2$
I_0	IV	IV^*	$\mathbb{Z}_3$
I_0	III	III^*	$\mathbb{Z}_4$
I_0	II	II^*	$\mathbb{Z}_6$

modular (3)			G
I_1	I_1	I_4^*	$\Gamma_0(4)$
I_2	I_2	I_2^*	$\Gamma(2)$
I_3	I_1	IV^*	$\Gamma_0(3)$
I_2	I_1	III^*	$\Gamma_0(2)$
I_1	I_1	II^*	Γ

modular (4)				G
I_4	I_2	I_2	I_4	$4E^0$
I_2	I_1	I_1	I_8	$\Gamma_0(8)$
I_3	I_3	I_3	I_3	$\Gamma(3)$
I_9	I_1	I_1	I_1	$\Gamma_0(9)$
I_5	I_1	I_1	I_5	$\Gamma_1(5)$
I_6	I_1	I_2	I_3	$\Gamma_0(6)$

$G \subset \mathrm{SL}(2,\mathbb{Z})$ generated monodromy group

Extremal rational surfaces and their periods

- Rational elliptic surfaces **S** with section

$$\bar{\mathbf{S}}: \; y^2 = 4\,x^3 - g_2\,x - g_3\,, \qquad \begin{matrix} g_2 \in H^0(\mathcal{O}(4)), \\ g_3 \in H^0(\mathcal{O}(6)), \end{matrix} \quad [t:1] \in \mathbb{CP}^1.$$

- Extremal rational surfaces (up to $*$-transfer):

modular (3)			μ
I_1	I_1	I_4^*	$1/2$
I_2	I_2	I_2^*	$1/2$
I_3	I_1	IV^*	$1/3$
I_2	I_1	III^*	$1/4$
I_1	I_1	II^*	$1/6$

modular (4)				a	q
I_4	I_2	I_2	I_4	-1	0
I_2	I_1	I_1	I_8	-1	0
I_3	I_3	I_3	I_3	$\frac{1-i\sqrt{3}}{2}$	$\frac{3-i\sqrt{3}}{2}$
I_9	I_1	I_1	I_1	$\frac{1-i\sqrt{3}}{2}$	$\frac{3-i\sqrt{3}}{2}$
I_5	I_1	I_1	I_5	$\dots$	$\dots$
I_6	I_1	I_2	I_3	$1/9$	$1/3$

Solutions to Picard-Fuchs rank-2 first order linear system:

$$\omega = {}_2F_1(\mu, 1-\mu; 1|t) \qquad \omega = Hl(a, q; 1, 1, 1, 1|t)$$

One-parameter families of K3 surfaces

- Construction 1: quadratic twist with polynomial h

$$\begin{array}{rcl} \bar{\mathbf{X}}_1 = \bar{\mathbf{S}}_h : \; Y^2 & = & 4\,X^3 - h^2\,g_2\,X - h^3\,g_3 \\ & \downarrow & \\ \bar{\mathbf{S}} : \; y^2 & = & 4\,x^3 - g_2\,x - g_3 \,. \end{array}$$

- Twist adds 2 fibers of type I_0^*
- Parameter defines position of additional I_0^*, $h = t(t-A)$
- 1-parameter families of lattice-polarized K3 surfaces ($\rho = 19$)
- Example: $\mathrm{T}_{\mathbf{X}} = \langle 2 \rangle^{\oplus 2} \oplus \langle -2 \rangle$, $A \notin \{0, 1\}$:

E_{sing}	I_2	I_2	I_2^*
t	0	1	∞

S is rational

E_{sing}	I_2^*	I_2	I_2^*	I_0^*
t	0	1	∞	A

$\mathbf{X}_1$ is K3

One-parameter families of K3 surfaces

- Construction 1: quadratic twist with polynomial h

$$\begin{array}{rcl}\bar{\mathbf{X}}_1 = \bar{\mathbf{S}}_h : \ Y^2 & = & 4\,X^3 - h^2\,g_2\,X - h^3\,g_3 \\ & \downarrow & \\ \bar{\mathbf{S}} : \ y^2 & = & 4\,x^3 - g_2\,x - g_3\,.\end{array}$$

- 2 I_0^*'s, $h = t(t-A)$, 2-form: $dt \wedge \frac{dX}{Y} = \frac{1}{\sqrt{h(t)}}\, dt \wedge \frac{dx}{y}$
- Represent K3-periods as **Euler transform** of a HGF
  $\Omega_{ij} = \oiint_{S_{ij}} dt \wedge \frac{dX}{Y} = \int_{t_i^*}^{t_j^*} dt\, \frac{1}{\sqrt{h(t)}}\, \omega$
- They solve a 3rd oder ODE (=symmetric square of 2nd order).

Solutions to the rank-3 integrable linear system of K3 periods:

$$\Omega = {}_3F_2\left(\begin{array}{c} \mu, \frac{1}{2}, 1-\mu \\ 1, 1 \end{array} \middle| A \right) \qquad \Omega = \left[\, HI\left(a, \tfrac{q}{4}; \tfrac{1}{4}, \tfrac{3}{4}, 1, \tfrac{1}{2} \middle| A \right) \right]^2$$

One-parameter families of K3 surfaces

- Construction 1: quadratic twist with polynomial h

$$\begin{array}{rcl} \bar{\mathbf{X}}_1 = \bar{\mathbf{S}}_h : \; Y^2 & = & 4\,X^3 - h^2\,g_2\,X - h^3\,g_3 \\ & \downarrow & \\ \bar{\mathbf{S}} : \; y^2 & = & 4\,x^3 - g_2\,x - g_3 \, . \end{array}$$

- 2 I_0^*'s, $h = t(t-A)$, 2-form: $dt \wedge \frac{dX}{Y} = \frac{1}{\sqrt{h(t)}}\, dt \wedge \frac{dx}{y}$
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  $\Omega_{ij} = \oiint_{S_{ij}} dt \wedge \frac{dX}{Y} = \int_{t_i^*}^{t_j^*} dt \, \frac{1}{\sqrt{h(t)}} \, \omega$
- They solve a 3rd oder ODE (=symmetric square of 2nd order).

Solutions to the rank-3 integrable linear system of K3 periods:

$$\Omega = \left[{}_2F_1\left(\tfrac{\mu}{2}, \tfrac{1-\mu}{2}; 1 \middle| A\right) \right]^2 \qquad \Omega = \left[Hl\left(a, \tfrac{q}{4}; \tfrac{1}{4}, \tfrac{3}{4}, 1, \tfrac{1}{2} \middle| A\right) \right]^2$$

One-parameter families of K3 surfaces

Proposition (M.-Doran)

- *There is a fundamental set of solutions* $\{x_1, x_2, x_3\}$ *such that*

μ	quadric surface	series
1/2	$x_1^2 + x_2^2 - x_3^2$ $2\,x_1^2 + 2\,x_2^2 - 2\,x_3^2$	${}_3F_2\left(\begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{array} \middle\vert A\right) = \sum_{n=0}^{\infty} \frac{(2n)!^3}{n!^6} \frac{A^n}{2^{6n}}$
1/3	$4\,x_1^2 + 3\,x_2^2 - 3\,x_3^2$	${}_3F_2\left(\begin{array}{c} \frac{1}{3}, \frac{1}{2}, \frac{2}{3} \\ 1, 1 \end{array} \middle\vert A\right) = \sum_{n=0}^{\infty} \frac{(2n)!\,(3n)!}{n!^5} \frac{A^n}{2^{2n}3^{3n}}$
1/4	$4\,x_1^2 + 2\,x_2^2 - 2\,x_3^2$	${}_3F_2\left(\begin{array}{c} \frac{1}{4}, \frac{2}{4}, \frac{3}{4} \\ 1, 1 \end{array} \middle\vert A\right) = \sum_{n=0}^{\infty} \frac{(4n)!}{n!^4} \frac{A^n}{4^{4n}}$
1/6	$x_1^2 + 4\,x_2^2 - x_3^2$	${}_3F_2\left(\begin{array}{c} \frac{1}{6}, \frac{3}{6}, \frac{5}{6} \\ 1, 1 \end{array} \middle\vert A\right) = \sum_{n=0}^{\infty} \frac{(6n)!}{n!^3(3n)!} \frac{A^n}{2^{6n}3^{3n}}$

- *First 4 cases with 4 singularities are obtained as double covers.*
- *Cases 5 and 6 are related to Apery's recurrence for* $\zeta(2)$ *and* $\zeta(3)$*:*

$$e.g.,\ \Omega = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \right) \frac{A^n}{4^n}$$

One-parameter families of K3 surfaces

- Construction 2: double cover branched at $t = 0$ and $t = A$:

$$\begin{array}{rcl} \bar{\mathbf{X}}_2 = \bar{\mathbf{S}}_{[0,A]} : \; Y^2 & = & 4\,X^3 - s^4\, g_2(t(s))\, X - s^6\, g_3(t(s)) \\ & \downarrow & \\ \bar{\mathbf{S}} : \; Y^2 & = & 4\,X^3 - g_2(t)\, X - g_3(t)\,. \end{array}$$

- with $t = \frac{(s+A/4)^2}{s}$ we have $ds \wedge \frac{dX}{Y} = \frac{1}{\sqrt{t(t-A)}}\, dt \wedge \frac{dx}{y}$
- Example of 1-param. family of lattice-polarized K3 surface of Picard rank 19, $\mathrm{T}_{\mathbf{X}} = H \oplus \langle -2 \rangle$, $A \notin \{0, 1\}$:

E_{sing}	I_1	I_1	II^*
t	0	1	∞

$\mathbf{S}$ is rational

E_{sing}	I_2	$2\,I_1$	$2\,II^*$
s	$A/4$	$\frac{A}{4} + \frac{1}{2} \pm \sqrt{A+1}$	$0, \infty$

$\mathbf{X}_2$ is K3

- $\mathbf{X}_2$'s are one-parameter families with $n = 1, 2, 3, 4, 5, 6, 8, 9$ and $M_n = H \oplus E_8 \oplus E_8 \oplus \langle -2n \rangle$ lattice polarization.

One-parameter families of K3 surfaces

Proposition (M.-Doran)

- *The two constructions give rise to degree-two rational maps* $\mathbf{X}_2 \dashrightarrow \mathbf{X}_1$ *that leave the holomorphic two-form invariant.*
- *The Picard-Fuchs differential equations of each pair* $\mathbf{X}_2, \mathbf{X}_1$ *coincide.*

Remarks:

- The periods of the families with M_n lattice polarization for $n = 1, 2, 3, 4, 6$ agree with the results of **Lian, Yau ['96]**, **Dolgachev ['96]**, **Verrill, Yui['00]**, **Doran ['00]**, and **Beukers, Stienstra, Peters ['84, '85, '86]**.
- One can "undo" the Kummer construction and provide interpretation of K3 periods in terms of modular forms:

$$
{}_2F_1\left(\frac{\mu}{2}, \frac{1-\mu}{2}; 1 \middle| A\right) = {}_2F_1\left(\mu, 1-\mu; 1 \middle| a\right), \; A = 4a(1-a),
$$
$$
Hl\left(a, \frac{q}{4}; \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2} \middle| A\right) \sim Hl\left(a, q; 1, 1, 1, 1 \middle| a\right), \; A = quartic(a).
$$

Two-parameter families of K3 surfaces

Set $h(t) = (t-A)(t-B)$ in $\mathbf{X}_1$ and $t = \frac{16\,s^2 + 8\,(A+B)\,s + (A-B)^2}{16\,s}$ in $\mathbf{X}_2$ s.t.

$$ds \wedge \frac{dX}{Y} = \frac{1}{\sqrt{h(t)}}\, dt \wedge \frac{dx}{y}$$

Proposition (M.-Doran)

- *The two constructions give rise to degree-two rational maps* $\mathbf{X}_2 \dashrightarrow \mathbf{X}_1$ *(for all cases with* $\kappa = 0$*) that leave the holomorphic two-form invariant.*
- *The Picard-Fuchs linear systems for each pair* $\mathbf{X}_2, \mathbf{X}_1$ *coincide.*
- *K3-periods solve an integrable rank-4 linear system in* ∂_A, ∂_B*.*
- *In cases with 3 singular fibers, solution is Appell HGF:*

$$\Omega_\mu(A,B) = \frac{1}{B^\mu}\, F_2\left(\mu; \frac{1}{2}, \mu; 1, 2\mu \Big| 1 - \frac{A}{B}, \frac{1}{B}\right)$$

Two-parameter families of K3 surfaces

Remarks:

- F_2 satisfies equations of a linear system of rank 4:

$$A(1-A)\,F_{AA} + A\,B\,F_{AB} + \Big(\gamma\ - (\alpha+\beta\ +1)\,A\Big)\,F_A - \beta\ \,B\,F_B - \alpha\,\beta\ F = 0\,,$$

$$B(1-B)\,F_{BB} + A\,B\,F_{AB} + \Big(\gamma' - (\alpha+\beta'+1)\,B\Big)\,F_B - \beta'\,A\,F_A - \alpha\,\beta' F = 0\,.$$

- Example ($\mu = 1/6$): $M = H \oplus E_8 \oplus E_8$-polarized case,

E_{sing}	$2\,I_1$	$2\,I_1$	$2\,II^*$
s	$t(s)=0$	$t(s)=1$	$0,\ \infty$

$\mathbf{X}_2$ is M-polarized K3

$\overset{s.l.}{\dashrightarrow}$

E_{sing}	$2\,I_1$	II^*	$2\,I_0^*$
t	$0,1$	∞	A, B

$\mathbf{X}_1$ is $\mathrm{Kum}(E_1 \times E_2)$

- Examples realize elliptic fibrations $\mathfrak{J}_3, \mathfrak{J}_4, \mathfrak{J}_6, \mathfrak{J}_7, \mathfrak{J}_{11}$ on $\mathrm{Kum}(E_1 \times E_2)$ from **Oguiso ['88]**.

Two-parameter families of K3 surfaces

Remarks:

- Ω's satisfy Quadratic Condition (cf. **Sasaski, Yoshida ['88]**): fundamental solutions (x_1, x_2, x_3, x_4) are quadrically related, solution surfaces $S \subset \mathbb{P}^3$ reduces to $\mathbb{P}^1 \times \mathbb{P}^1$.
- Clausen-type equation:

$$\frac{1}{B^\mu} F_2\left(\mu; \tfrac{1}{2}, \mu; 1, 2\mu \middle| 1 - \tfrac{A}{B}, \tfrac{1}{B}\right) = \frac{1}{(1-A-B)^\mu} \cdot \begin{array}{c} {}_2F_1\left(\frac{\mu}{2}, \frac{\mu+1}{2}; 1 \middle| x\right) \\ {}_2F_1\left(\frac{\mu}{2}, \frac{\mu+1}{2}; \mu + \frac{1}{2} \middle| y\right) \end{array}$$

where $x(1-y) = \left(\frac{A-B}{1-A-B}\right)^2$, $y(1-x) = \left(\frac{1}{1-A-B}\right)^2$.

- F_2 satisfies linear and quadratic transformations (symmetries)
linear: $\Omega_\mu(A, B) = \Omega_\mu(B, A)$

Two-parameter families of K3 surfaces

Remarks:

- F_2 satisfies linear and quadratic transformations (symmetries)

  quadratic: $\Omega_{1/2}(A,B) = \left(\dfrac{2\,B}{1-A-B}\right)^{1/2} \Omega_{1/4}(\tilde{A},\tilde{B})$

  $$\text{with } \tilde{A} = \left(\tfrac{A-B+1}{A+B-1}\right)^2,\ \tilde{B} = \left(\tfrac{A-B-1}{A+B-1}\right)^2.$$

- If we specialize $A = (\lambda/4)^2, B = 1 + A$ then we obtain

  $$\Omega_{1/2}(A,B) = \Omega_{1/4}\left(0, \left(\tfrac{4}{\lambda}\right)^4\right) = {}_3F_2\left(\begin{matrix}\frac{1}{4},\frac{2}{4},\frac{3}{4}\\ 1,1\end{matrix}\,\middle|\,\left(\tfrac{4}{\lambda}\right)^4\right)$$

  for the period of the sub-family $\mu = \frac{1}{2}$ (agrees with **Narumiya, Shiga ['01]**) which is birational to

  $$\mathcal{F} = \left\{xyz\,(x+y+z+\lambda)+1=0\right\} \subset \mathbb{P}^3\,,$$

Periods of 3-parameter families of K3 surfaces

- There is only one family where the construction of $\mathbf{X}_1$ can be turned into a 3-parameter family of K3 surfaces with lattice polarization of Picard rank 17: SW-curve $\mu = 1/2$.
- Use $h(t) = (t-A)(t-B)(t-C)$ to obtain linear system of rank 5 in A, B, C for the K3-periods on $\mathbf{X}_1$ = specialization of Aomoto-Gel'fand HGF of type (3,6)

$$E(3,6)\left(\alpha_i = \frac{1}{2}\middle| u, v, 0, w\right)$$

where $u = \left(\frac{C-A}{B-A}\right)\frac{B}{C}, v = \frac{B}{C}, w = B.$

- Linear system specialization of the one in **Matsumoto et. al ['93]** for a family of K3 surfaces of Picard rank 16 associated with six lines in the complex plane, no three of which are concurrent.

Kummer surfaces from $SU(2)$-Seiberg-Witten curve

Proposition (M.-Doran)

- *The family* $\mathbf{X}_1 = \mathbf{S}_h \to \mathbb{CP}^1$ $(\mu = 1/2)$ *is a family of Jacobian K3 surfaces of Picard rank 17.*
- *There is a family* $\mathbf{X}_2 \to \mathbb{CP}^1$ *obtained from the covering map* $t = (C\,s^2 - B)/(s^2 - 1)$.
- $\mathbf{X}_2 = \mathrm{Kum}(\mathbf{A})$ *where*

ρ	parameter	$\mathbf{A}$	equation	moduli space
17	u, v, w	$\mathrm{Jac}\, C^{(2)}$	$y^2 = x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$	$\Gamma_2(2)\backslash\mathbb{H}_2$
18	$u, w, v=0$	$E_1 \times E_2$	$y_i^2 = (2x_i - 1)\left[(4x_i+1)^2 + 9r_i\right]$	$\Gamma\backslash\mathbb{H}\times\mathbb{H}$
19	$u, v=0, w=1$	$E_1 \times E_1$	$y_1^2 = (2x_1 - 1)\left[(4x_1+1)^2 + 9r_1\right]$	$\Gamma_0(2)\backslash\mathbb{H}$

Mayr, Stieberger ['95], Kokorelis ['99]: moduli space of genus-two curves with level-two structure = moduli space of $\mathcal{N} = 2$ heterotic string theories compactified on $K3 \times T^2$ with one Wilson line.