CHARACTERISTIC CLASSES, Γ-FUNCTIONS, σ-MODELS

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E = COMPLEX V.B. $\quad \pi^{-1}(U) \cong U \times \mathbb{C}^N$

$\pi \downarrow$

M = COMPACT SMOOTH MFLD $\quad$ TRANS. FUNC. IN $GL(N, \mathbb{C})$

A CONNECTION ON E IS A WAY OF DIFF'ING LOCAL SEC'NS

$$\nabla : E \to \cancel{E \otimes \Omega_X^1}\ \Omega_X^1 \otimes E$$

$s \in \Gamma(U, E)$ $\quad$ (U = open set)

$f \cdot s \in \Gamma(U, E) \qquad f$ = function

LEIBNIZ RULE $\quad \nabla(fs) = df \otimes s + f\,\nabla(s)$

CURVATURE $\quad F^\nabla \in \Gamma(X, \Omega_X^2 \otimes \mathrm{End}(E))$

$\mathfrak{gl}(N, \mathbb{C})$ locally

$$F^\nabla_{(s)}(X, Y) = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s$$

TOPOLOGICAL INVT'S

WANT: ~~INVT~~ POLYNOM. FUNCTIONS ON $\mathfrak{gl}(N, \mathbb{C})$ INVT UNDER CONJ $\quad f(g^{-1} z g) = f(z) \qquad z \in \mathfrak{gl}(N, \mathbb{C}),\ g \in GL(N, \mathbb{C})$

$f(F^\nabla)$ = AN EVEN DIFF FORM ON X

LEMMA (CHERN) $\quad df(F^\nabla) = 0$

$$[f(F^\nabla)] \in H^{even}(X, \mathbb{C})$$

$$c(E) = c(F^\nabla) = \det\left(I + \tfrac{i}{2\pi} F^\nabla\right)$$

$$= 1 + c_1(F^\nabla) + c_2(F^\nabla) + \ldots$$

$c_j(E) \in H^{2j}(X_j, \mathbb{Z}) / \text{tors.} \qquad c_j(E) \to [c_j(F^\nabla)]$

DIFF F^∇ HAVE SAME $[c_j(F_1^\nabla)] = [c_j(F_2^\nabla)] = c_j(E)$

IF Y KÄH. MFLD E A HERM BUNDLE

$\Rightarrow$ $\exists$ CONNECTION ST $c_j(F^\nabla) \in H^{j,j}(X)$

IF X KÄH,
X HAS A "HOLOM TANGENT BUNDLE" $T_X^{1,0}$

$c_j(T_X^{1,0}) =: c_j(X)$

<u>HIRZEBRUCH - RIEMANN-ROCH</u>

~~IF~~ V.B. E

$$\begin{aligned} &\dim H^0(X,E) \\ &- \dim H^1(X,E) \\ &+ \dim H^2(X,E) \\ &=: \chi(X,E) \\ &\quad \| \text{ H.R.R.} \\ &\int_X ch(E) \wedge td_X \end{aligned}$$

ASIDE
$$ch(E) = [ch(F^\nabla)] = tr\left(\exp\left(\tfrac{i}{2\pi}F^\nabla\right)\right)$$

<u>SPLITTING PRINCIPLE</u> (Grothendieck)
WE MAY PRETEND

$$E = L_1 \oplus \dots \oplus L_N \qquad rk(L_j) = 1 \qquad N = rk(E)$$

$$c(E) = \prod (1 + c_1(L_j)) = \sum c_j(E) \qquad \lambda_j = c_1(L_j)$$

$$ch(E) = \sum e^{c_1(L_j)}$$

<u>MULTIPLICATIVE CHAR. CLASSES</u>

<u>INGRED</u>: POWER SERIES $p(z)$ WITH CONST TERM 1

~~PK~~ <u>METHOD</u>: $\prod(p, c_1(L_j))$

$p(z) = 1 + z_i \qquad c(E) = \prod(1 + c_1(L_j))$

$td_X \longleftrightarrow \frac{z}{1-e^{-z}}$ NB $td_X = e^{c_1(X)/2} \hat{A}_X$

$\hat{A}_X \longleftrightarrow \frac{z/2}{\sinh(z/2)}$

MUKAI: REALIZED $\mathcal{E}, \mathcal{F}$ = coh. sheaves on alg variety

$\mathrm{HOM}(\mathcal{E}, \mathcal{F}) \ \ldots \ \mathcal{E}xt^i(\mathcal{E}, \mathcal{F}) \ldots$

$\chi(\mathcal{E}, \mathcal{F}) = \sum (-1)^i \dim \mathrm{Ext}^i(\mathcal{E}, \mathcal{F})$

ASIDE 2

$ch: K(X) \to H^{ev}(X, \mathbb{Z})$

IS AN ISOMORPHISM AFTER $\otimes \mathbb{Q}$

ch $K(X)$, define pairing by

$\langle \mathcal{E} | \mathcal{F} \rangle \equiv \chi(\mathcal{E}, \mathcal{F}) = \int_X ch(\mathcal{E}^\vee) \wedge ch(\mathcal{F}) \wedge td_X$

$\hookrightarrow$ DEFINE PAIRING ON K-THEORY VIA THE ISOMORPHISM

$\widetilde{ch(E)} = ch(E) \wedge \sqrt{td_X}$

$v^\vee = \frac{\tau(v)}{\sqrt{ch(\omega_X)}}$ WHERE $\tau(v)$ MULT BY $(\sqrt{-1})^k$ ON $H^k(X)$

$\langle v | w \rangle = \int_X v^\vee \wedge w$

$\frac{\pi w}{\sin \pi w} = \Gamma(1+w)\Gamma(1-w)$

$\sqrt{td_X}$ or $\sqrt{\hat{A}_X}$ USE $\Gamma(1+z)$ AS A MULT. CHAR CLASS

SOMETIMES $\Gamma\left(1+\frac{iz}{\pi}\right)$

CALL RESULTING CLASS Γ_X

PT: (DAVE'S MOTIVATION)

VARIATIONS OF HODGE STRUCT $\xleftrightarrow{MS}$ VARIATIONS IN H^{ev}, Γ CLASS APPEARS

$\Lambda(z) \equiv \mathrm{Im}\log\Gamma\left(1+\frac{z}{2\pi i}\right) = \frac{\gamma}{2\pi}z + \sum(-)^k \frac{\zeta(2k+1)}{2k+1}\left(\frac{z}{2\pi}\right)^{2k+1}$

$\hat{\Lambda}(z) = \Lambda(z) - \frac{\gamma}{2\pi}z$

$\Gamma_X = \sqrt{\mathrm{Td}_X} \wedge e^{i\Lambda}$

SUSY ON WS $\longleftrightarrow$ KAHLER TARGET

RENORMALIZATION OF $\mathcal{N}=(2,2)$ SUSY NLSM'S IN 1+1 D

$\Sigma \to X$ EUCLIDEAN THEORY

ACTION DEPENDS ON KÄH. MET. ON X

RENORMALIZATION: MEASURES RESPONSE TO SCALE CHANGE

"β-function" of theory DEPENDS ON COUPLING CONSTANT α'

β VANISHES TO ZEROTH ORDER IN α'

$\frac{1}{\alpha'}\beta_{ij} = \mathrm{Ric}_{ij} + \alpha'(\ldots)$

$= \mathrm{Ric}_{ij} + 0\,\alpha' + 0\,(\alpha')^2 + \frac{\zeta(3)}{48}T_{ij}(\alpha')^3 + 0\,(\alpha')^4 + \ldots$

CONJECTURED ANSWER $e^{-i\Lambda(\tau)} = \zeta(3)c_3 + \zeta(5)c_5 - \zeta(3)c_3c_2 + \ldots$

CANDELAS HOROWITZ STROMINGER WITTEN